Journal of Pure and Applied Algebra 19 (1980) 347–383

VARIETIES OF CHAIN-COMPLETE ALGEBRAS*

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1. Introduction

This paper is an extended version of results and ideas in [78] and [79]. Several of the properties given here for categories of algebras have been previously studied for their underlying base categories in [80] and [82].

The main purpose of this work is to show how the point of view of Lawvere about classical Universal Algebra: "equational theories are categories, algebras are functors, homomorphisms are natural transformations," provides an optimal conceptual and technical tool for developing the kind of generalized Universal Algebra which is intimately related with the semantics of programming languages.

Any choice of historical references in an area is always difficult, but we think that papers like Eilenberg and Wright [38], Scott [94] and Bekić [13] are among those at the beginning of what might be called the Universal Algebra approach to Computation. The list of references can give some indication of the work which has followed, but for a far more complete orientation one should consult the bibliography collected in [10].

Due perhaps to an emphasis in monadic computation, where predicates $p : A \to 2$ can be viewed as functions

$$\tilde{p} : A \to A \times \{0\} \cup A \times \{1\} = A \amalg A \; : x \mapsto (x, p(x)),$$

several of the approaches which make an explicit use of algebraic theories in connection with program schemas, have favoured a coproduct-preserving semantics, and have been primarily concerned with theory morphisms, rather than with the relationship between theories and algebras, as a consequence. As we mentioned in [78], the "if then else" use of predicates: $(p \to -, -) : A^3 \to A : (x, y, z) \mapsto$ if $p(x)$ *then* y *else* z, as in Nivat [83], is the one which provides the key link between theories and algebras.

The (graph of) the partial function f defined by a recursive program is the union

* Text of a lecture given May 26 1979, at Aspen, Colorado, in honor of Saunders MacLane's seventieth birthday.

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or "limit" of the ascending chain $\{f_n \mid n \in \omega\}$, where f_n is the portion of f which can be computed with at most n nested recursive calls. The order-theoretic approach to computation introduced by Scott [93], allows the necessary elbow space to encompass the basic objects and functions as well as the limits of both, which appear naturally associated to recursive definitions. By bringing finite and infinite elements under the same roof, it also allows abstract syntax and infinite "free" computations. The choice of the right category of complete posets and continuous (i.e. limit-preserving) functions is to a good extent a matter of controversy. Following other authors, we shall adopt here a maximalist solution, **Pos** (ω), which hardly could be wrong because of allowing too little room, but still has enough good properties: objects will be (ω)-chain-complete posets (i.e. any countable ascending chain $\{a_n\}$ has a limit) and maps will be continuous functions, i.e. monotonic functions which preserve limits of chains. A closely related category **Pos** $(\dot{\omega})$, which contains **Pos** (ω) as full reflective subcategory is also introduced. We find it useful to have the flexibility of concentrating the attention, not in all the limits of all the chains, but in those which are relevant to each situation. Many of the results that we give can however be generalized to a family of categories of the form **Pos** (Z), for Z an adequate parameter (see [7, 82] and the comments in Sections 2 and 3), which share most of the properties of the instances **Pos** (ω) and **Pos** $(\dot{\omega})$.

Section 2 contains the basic definitions and properties of the categories **Pos** (posets and monotonic functions), **Pos** (ω) and **Pos** $(\dot{\omega})$. Some of the proofs, which can be found in [80] and [82], are omitted, but enough detail is given as to make the paper reasonably self-contained.

Section 3 gives results and constructions similar to those of classical Universal Algebra for the "classical" (i.e. corresponding to ordinary Lawvere theories) varieties of algebras on the base categories **Pos**, **Pos** (ω) and **Pos** $(\dot{\omega})$.

Section 4 studies the varieties that are obtained, when ordinary theories are replaced by theories enriched in any of the three categories in question, shows how the results of Lawvere extend to this context, and proves a variety theorem similar to the classical one of Birkhoff.

Section 5 develops from scratch the relationship between chain-complete algebras and interpretations of a programming language, shows the relationship between varieties and classes of interpretations, and indicates how varieties on the category **Pos** $(\dot{\omega})$ can be used to connect regular algebras and rational theories. More detailed introductions are given at the beginning of each section.

This paper is respectfully dedicated to Professor Saunders MacLane. His work and his ideas have exerted a strong influence in my formation, from the beginning of my graduate years, as have done too his person and example.

I want to express my gratitude to Joseph Goguen, Klaus Indermark and Michael Pfender, from whom I have learned many ideas in Semantics and Universal Algebra, along many conversations, and to Bruno Courcelle, John Gray, F. William Lawvere, Daniel Lehmann, Maurice Nivat, Dana Scott, Adrian Tang, James Thatcher, Eric Wagner and Jesse Wright for their valuable comments,

intellectual generosity, and the most encouraging stimulus of their work. Special thanks go to Ignacio Sols friend and co-worker for many years, for his careful reading of the manuscript and many suggested improvements.

The results from Category Theory which are used can be found in the book of MacLane [70], and, for basic properties of factorization systems, in Herrlich and Strecker [54], except for the background on algebraic theories for which one may consult the original papers of Lawvere [61, 62, 63] and Bénabou [14] as well as [2, 38, 85, 91]. The basic definitions for V-categories (V a closed symmetric monoidal category) can be found in Dubuc [34].

2. Chain-complete posets

For the convenience of the reader we gather in this section the results on the ground categories of chain-complete posets that we shall use in the rest of the paper to obtain similar constructions for algebras. The exposition is somewhat condensed, but full details can be found in [80] and [82]. By motivational and expository reasons, we stick to the denumerable chain-complete case but, except for local presentability, which expresses the first-order character of denumerable chain-completeness, most of the properties we mention have been proved in [82] for any crossed-down subset system in the sense of [7].

A *partially ordered set* or *poset* is a set A, together with a reflexive, transitive and antisymmetric relation $\leq$ on A. A function $f: A \to B$ is *monotonic* if it is order-preserving, i.e. $a, a' \in A,\ a \geq a' \Rightarrow fa \geq fa'$. In the sequel, **Pos** will denote the category of posets and monotonic functions. A poset A is called ω-*(chain) complete* if every denumerable ascending chain in A : $\{a_n\} \subseteq A,\ a_{n+1} \geq a_n\ n \in \omega$, has a least upper bound, denoted $\bigsqcup a_n$. A poset is called $\dot{\omega}$-*(chain) complete,* if every denumerable ascending chain $\{a_n\}$ in A which is bounded in A (i.e. $\exists a \in A,\ a \geq a_n\ \forall n \in \omega$), has a least upper bound. A monotonic map $f: A \to B$ between two posets is ω-continuous, if for any denumerable chain $\{a_n\}$ in A such that $\bigsqcup a_n$ exists in A, then $\bigsqcup f(a_n)$ exists in B, and it is $f(\bigsqcup a_n) = \bigsqcup f(a_n)$. $\dot{\omega}$-complete posets and ω-continuous maps form a category that we shall denote **Pos** $(\dot{\omega})$. ω-complete posets and ω-continuous maps form then a full subcategory of **Pos** $(\dot{\omega})$, denoted **Pos** (ω). Hence we have inclusion functors

$$\mathbf{Pos}\ (\omega) \hookrightarrow \mathbf{Pos}\ (\dot{\omega}) \hookrightarrow \mathbf{Pos}$$

For simplicity of terminology, in the sequel *chain* or ω-*chain* will mean a denumerable ascending chain; a bounded ω-chain will sometimes be called a $\dot{\omega}$-*chain,* and ω-continuous can be shortened to *continuous.*

The set of monotonic functions between two posets A, B, has a poset structure, denoted $[A, B]$, in the obvious way: $f \geq g$ iff $\forall a \in A,\ fa \geq ga$. Similarly, the set of continuous functions between two ω-complete posets is a ω-complete poset, denoted $\omega[A, B]$, and the set of continuous functions between two ω-complete posets is a $\dot{\omega}$-complete poset denoted $\dot{\omega}[A, B]$. These "internal hom functors"

$[-,-]$, $\omega[-,-]$, $\dot{\omega}[-,-]$, together with the direct product of posets (componentwise ordering) make **Pos**, **Pos** (ω) and **Pos** $(\dot{\omega})$ cartesian closed. It is immediate to check that **Pos** (ω) and **Pos** $(\dot{\omega})$ are complete, and that the limits in these categories are the ones computed in **Pos**.

Pos has a coequalizer-mono and an epi-equalizer factorization. A map $m: A \to B$ in **Pos** is an equalizer iff it is a *full monomorphism,* i.e. iff $\forall a,\ a' \in A$, $ma \geq ma' \Leftrightarrow a \geq a'$, and it is a monomorphism iff it is injective. A map is an epi iff it is surjective. In **Pos** (ω) and **Pos** $(\dot{\omega})$, the set-theoretic image is not ω-complete (resp. $\dot{\omega}$-complete) in general, but there are several factorization systems that we shall now consider. As in both, monomorphisms coincide with injective maps, they are well-powered, and have (see [54]) extremal epi-mono and epi-extremal mono factorization systems. It is shown in [80] for **Pos** (ω), but can also be easily adapted to **Pos** $(\dot{\omega})$, that the class of extremal epis contains properly the one of coequalizers and the class of extremal monos contains properly the one of equalizers. Given an object A in **Pos** (ω), the subobjects which are full monomorphisms are closed under arbitrary intersections, and then taking for any $f: A \to B$ the full subobject $\overline{fA} \xrightarrow{m} B$, which is the intersection of all full subobjects through which f factors, we get a factorization $f = A \xrightarrow{e} \overline{fA} \xrightarrow{m} B$, corresponding to a factorization system that we shall call *strongly dense-full mono* (see (80) for more details, but notice that there, strongly dense maps are called dense). $\overline{fA}$ can be constructed by transfinite induction as: $\overline{fA} = \bigcup_\alpha fA_\alpha$; $fA_0 = fA$, $fA_{\alpha-1} = fA_\alpha \cup \{\bigsqcup a_n \mid \{a_n\}$ chain in $fA_\alpha\}$, for α limit ordinal, $fA_\alpha = \bigcup_{\beta<\alpha} fA_\beta$. Of course f is strongly dense iff $\overline{fA} = B$. It is not known to the author (but seems very plausible) if full monomorphisms coincide with extremal monos[1] in **Pos** (ω), (then epis would coincide with strongly dense maps). In **Pos** $(\dot{\omega})$ there is similarly a strongly dense-full mono factorization system, but for $f: A \to B$ in **Pos** (ω), its strongly dense-full mono factorizations in **Pos** (ω) and **Pos** $(\dot{\omega})$ do not coincide in general. Actually, there is another factorization system in **Pos** $(\dot{\omega})$ such that, when restricted to objects in **Pos** (ω) gives the strongly dense-full mono factorization of **Pos** (ω). We call it *dense-persistently complete* factorization. A full monomorphism $m: A \to B$ in **Pos** $(\dot{\omega})$ is persistently complete (p.c.) if any chain in A whose image is bounded in B is bounded in A (hence has a limit in A). Persistently complete subobjects are closed under intersections and we get as above a factorization $f = A \xrightarrow{e} \overline{fA}' \xrightarrow{m} B$, with $\overline{fA}' \xrightarrow{m} B$ the intersection of all p.c. subobjects through which f factors. Again we have a factorization system, and $\overline{fA}'$ can be obtained as: $\overline{fA}' = \bigcup_\alpha fA_\alpha$, with $fA_0 = fA$, $fA_{\alpha+1} = fA_\alpha \cup \{\bigsqcup a_n \mid \{a_n\}$ chain in fA_α and bounded in $B\}$, for α a limit ordinal, $fA_\alpha = \bigcup_{b<\alpha} fA_\beta$. $f: A \to B$ dense means $\overline{fA}' = B$.

In the category **Pos**, the lattice of quotients of an object A is anti-isomorphic to the lattice of preorders (i.e. symmetric and transitive relations) containing the order $\geq A$, in the same way that in the category of sets quotients are anti-isomorphic to

[1] *Added in proof.* This question has been recently answered in the negative by Lehmann and Pasztor, "On a conjecture by Meseguer," Technion Comp. Sci. TR # 170, February 1980.

equivalence relations. In other words, if $q : A \to B$ is surjective and $Rq := (q^2)^{-1}(\geq B)$ is the preorder induced by q, then any $f : A \to C$ induces a monotonic $\bar{f}$ with $\bar{f} \cdot q = f$ iff $R_f \supseteq R_q$, and then $\bar{f}$ is unique. Conversely, a preorder $R \supseteq (\geq A)$ induces a quotient $q_R : A \to A/R$ in the obvious way, with $R_{q_R} = R$. For **Pos** (ω) and **Pos** $(\dot{\omega})$ there is no hope (see [80]) for a similar anti-isomorphism for the set of *all* quotients. Roughly speaking, too much detail may be hidden in the part of the codomain off the set-theoretic image, which is not describable from the domain. However, each has a subclass of quotients (the "nice" quotients) which exactly correspond to the adequate kind of preorders:

If $A \in$ **Pos** (ω), a surjective monotonic map $f : A \to B$ is continuous iff R_f satisfies

(*) If $a \in A$, $\{a_n\} \subseteq A$ is a chain, and $a \, R_f \, a_n \; \forall n \in \omega$, then $a \, R_f \bigsqcup a_n$.

If $A \in$ **Pos** $(\dot{\omega})$, a surjective monotonic map $f : A \to B$ is continuous iff R_f satisfies

(**) If $a \in A$, $\{a_n\} \subseteq A$ is a $\dot{\omega}$-chain, and $a \, R_f \, a_n \; \forall n \in \omega$, then $a \, R_f \bigsqcup a_n$.

For $A \in$ **Pos** (ω) (resp. $A \in$ **Pos** $(\dot{\omega})$) call a preorder $R \supseteq (\geq A)$ *ω-continuous* (resp. *$\dot{\omega}$-continuous*) iff it satisfies (*) (resp. (**)). ω-continuous preorders (resp. $\dot{\omega}$-continuous ones) are then closed under arbitrary intersections, and form a complete lattice. In other words: There is a closure operation on the lattice of binary relations on A, $Q \mapsto \bar{Q}$, which sends each relation to the smallest ω ($\dot{\omega}$)-continuous preorder containing it. The following then holds:

2.1. Lemma. *Given $A \in$ **Pos** (ω) (resp. $A \in$ **Pos** $(\dot{\omega})$) and R ω-continuous preorder (resp. $\dot{\omega}$-continuous preorder), there is an epimorphism $\tilde{q}_R : A \to (\widetilde{A/R})$ in **Pos** (ω) (resp. **Pos** $(\dot{\omega})$) with $R_{qR} = R$, and such that given a map $f : A \to B$ in **Pos** (ω) (resp. **Pos** $(\dot{\omega})$) there exists a continuous $\bar{f} : (\widetilde{A/R}) \to B$ such that $\bar{f} \cdot \tilde{q}_R = f$ iff $R_f \supseteq R$, and then $\bar{f}$ is unique.*

Proof. See [31] and [82]. □

Call now an epimorphism $f : A \to B$ in **Pos** (ω) (resp. **Pos** $(\dot{\omega})$) *nice* iff the induced continuous $\bar{f} : (\widetilde{A/R_f}) \to B$ is an isomorphism. Then we have an anti-isomorphism of the lattice of ω-continuous (resp. $\dot{\omega}$-continuous) preorders on A, and the lattice of its nice quotients. Coequalizers are a special case of nice epis. Any preorder R defines an equivalence relation E_R by: $a \, E_R \, a' \Leftrightarrow a \, R \, a'$ and $a' \, R \, a$. If R is ω or $\dot{\omega}$-continuous, E_R is just the (underlying set of the) kernel-pair of $\tilde{q}_R$. As any coequalizer is a coequalizer of its kernel-pair, it then follows that a map in **Pos** (ω) (resp. **Pos** $(\dot{\omega})$) is a coequalizer iff it is nice, and its corresponding continuous preorder R satisfies the condition

(2.1.1) $R = \bar{E}_R$.

Of course, if we start with an arbitrary *symmetric* relation Q, the preorder $R = \bar{Q}$ will satisfy (2.1.1). Given $f, g : A \to B$ in **Pos** (ω) (resp. **Pos** $(\dot{\omega})$), their coequalizer is then the nice epi corresponding to the ω (resp. $\dot{\omega}$)-continuous preorder generated by the relation $Q = \{(fa, ga), (ga, fa) \mid a \in A\}$.

As in **Pos** (ω) and **Pos** ($\dot{\omega}$) the coproduct is the disjoint union of sets with order the disjoint union of orders, both categories are cocomplete. They are also ω_1-*locally presentable* [42] (**Pos** is ω-locally presentable). This has been shown in [80] for **Pos** (ω), but only minor changes in the arguments given there will yield the result for **Pos** ($\dot{\omega}$).

Nice epis, coequalizers and factorizations in **Pos** (ω) and **Pos** ($\dot{\omega}$) are stable under products in the following sense:

2.2. Lemma. *For* **Pos** (ω) *and* **Pos** ($\dot{\omega}$), *if* $f_i : A_i \to B_i$, $i = 1, \ldots, n$ *are arbitrary morphisms and* $A_i \xrightarrow{e_i} C_i \xrightarrow{m_i} B_i$ $i = 1, \ldots, n$, *are their extremal epi-mono, resp. strongly dense-full mono, resp. (for* **Pos** ($\dot{\omega}$)) *dense-p.c. mono, resp. epi-extremal mono factorizations, then*

$$\prod A_i \xrightarrow{\prod e_i} \prod C_i \xrightarrow{\prod m_i} \prod B_i$$

is the factorization, of the respective type mentioned above, for $\prod_i f_i$.

If the maps f_i *are all nice epis (resp. coequalizers), then* $\prod f_i$ *is also a nice epi (resp. coequalizer).*

Proof. See [82], 4.9, 4.11, 4.12. □

The categories **Pos** (ω) and **Pos** ($\dot{\omega}$) are cowellpowered. A short proof has been pointed out by Lehmann [64], using the fact that $2 = \{0, 1\}$, with order $0 \leq 1$ is a cogenerator in both **Pos** (ω) and **Pos** ($\dot{\omega}$). For any of the classes of epis $\mathscr{E}$ in the factorization systems that we have considered, one then can show easily (cf. for instance [80]) that for any object A the set of $\mathscr{E}$-quotients forms a complete lattice. Note that nice epi quotients and coequalizer quotients of an object form also complete lattices, anti-isomorphic to the ones of continuous preorders and canonical kernel-pairs.

The inclusions **Pos** $(\omega) \hookrightarrow$ **Pos**, **Pos** $(\dot{\omega}) \hookrightarrow$ **Pos** have left adjoints or completion functors that we shall denote **Pos** $\xrightarrow{\wedge}$ **Pos** (ω), **Pos** $\xrightarrow{\dot{\wedge}}$ **Pos** ($\dot{\omega}$), and which can be easily described as follows: for any poset A, call $\Delta_\omega(A)$ to the set of countable directed subsets of A, and $\dot{\Delta}_\omega(A)$ to the subset of $\Delta_\omega(A)$ formed by those which are bounded. For any subset $S \subseteq A$ define $S{\downarrow} := \{x \in A \mid \exists y \in S \text{ such that } x \leq y\}$. Then

$$\hat{A} = \{S{\downarrow} \subseteq A \mid S \in \Delta_\omega(A)\}, \quad \text{ordered by inclusion,}$$
$$\dot{\hat{A}} = \{S{\downarrow} \subseteq A \mid S \in \dot{\Delta}_\omega(A)\}, \quad \text{ordered by inclusion.}$$

This follows from the fact that Δ_ω and $\dot{\Delta}_\omega$ are union-complete subset systems [7]. The units $\hat{\eta}A : A \to \hat{A}$, $\dot{\hat{\eta}}A : A \to \dot{\hat{A}}$, both defined by: $a \mapsto \{a\}{\downarrow}$, are clearly full monomorphisms. The inclusion **Pos** $(\omega) \hookrightarrow$ **Pos** ($\dot{\omega}$) has also a left adjoint, denoted **Pos** ($\dot{\omega}$) $\xrightarrow{\sim}$ **Pos** (ω), which is constructed in [82], 3.8–3.10. By inspection of the details given there, one has that the unit $\tilde{\eta}A : A \to \tilde{A}$, is a full monomorphism and a dense map in **Pos** ($\dot{\omega}$). These three completions preserve products (see [82], 4.8), hence we have natural isomorphisms: $A \mathbin{\hat{\times}} B \simeq \hat{A} \times \hat{B}$, $A \mathbin{\dot{\hat{\times}}} B \simeq \dot{\hat{A}} \times \hat{B}$, A, $B \in \mathbf{Pos}$; $A \mathbin{\tilde{\times}} B \simeq \tilde{A} \times \tilde{B}$, A, $B \in \mathbf{Pos}\,(\dot{\omega})$. Furthermore, the three inclusions are *monadic* (see [82], 3.5 and 3.13).

We finish this section with a lemma and corollary which will be useful in the sequel:

2.3. Lemma. *Let $e : A \to B$ be a dense map in* $\mathbf{Pos}\,(\dot{\omega})$ *(resp. strongly dense map in* $\mathbf{Pos}\,(\omega)$*). Then for any $C \in \mathbf{Pos}\,(\dot{\omega})$ (resp. $C \in \mathbf{Pos}\,(\omega)$) the map $\dot{\omega}[e, C] : \dot{\omega}[B, C] \to \dot{\omega}[A, C]$ is a p.c. monomorphism (resp. the map $\omega[e, C] : \omega[B, C] \to \omega[A, C]$ is a full monomorphism).*

Proof. By the relationship mentioned before between dense-p.c. mono factorizations in $\mathbf{Pos}\,(\dot{\omega})$, and strongly dense-full mono factorizations in $\mathbf{Pos}\,(\omega)$, it is enough to prove the case $\mathbf{Pos}\,(\dot{\omega})$.

The proof is by transfinite induction, using the generation process for $\overline{eA} = B$. We will prove first that $\dot{\omega}[e, C]$ is full monomorphism, i.e., given continuous $f, g : B \to C$, such that $fe \geq ge$, we are to see $f \geq g$, or equivalently $f|eA_\alpha \geq g|eA_\alpha$ for any ordinal α. For $\alpha = 0$ the result obviously holds, and for α a limit ordinal it also holds, if it does for any $\beta < \alpha$. Suppose now $f|eA_\alpha \geq g|eA_\alpha$, and let $\{a_n\} \subseteq eA_\alpha$ be a chain which is bounded in B. Then $f \bigsqcup a_n = \bigsqcup fa_n \geq \bigsqcup ga_n = g \bigsqcup a_n$. Hence $f|eA_{\alpha+1} \geq g|eA_{\alpha+1}$, and we have $\dot{\omega}[e, C]$ full monomorphism. To see that it is persistently complete, suppose that $\{f_n\} \subseteq \dot{\omega}[B, C]$ is a chain, and $\{f_n e\}$ is bounded in $\dot{\omega}[A, C]$. Then we have the continuous map $\bigsqcup f_n e$, which is defined by (cf. [82], 4.1) $(\bigsqcup f_n e)a := (\bigsqcup f_n\, e(a)$, $a \in A$. Hence on $e(A)$ we can define a function f by: $f y := \bigsqcup f_n\, y$, $y \in e(A)$, which is continuous because

$$f\Big(\bigsqcup_m y_m\Big) = \bigsqcup_n f_n\Big(\bigsqcup_m y_m\Big) = \bigsqcup_n \bigsqcup_m f_n y_m = \bigsqcup_m f y_m.$$

The function f can then be extended by transfinite induction to a continuous $f : B \to C$ such that $f = \bigsqcup f_n$ (which proves the lemma) because it can be extended to $e(A)_\alpha$, for α a limit cardinal, if it can to $e(A)_\beta$ for all $\beta < \alpha$, and if f can be extended to $e(A)_\alpha$ and $\{a_m\} \subseteq e(A)_\alpha$ is a chain bounded in B, we can define $f(\bigsqcup a_m) := \bigsqcup_n f_n \bigsqcup_m a_m = \bigsqcup_m f a_m$, which gives the desired continuous extension of f to $e(A)_{\alpha+1}$. $\square$

2.4. Corollary. *Given maps $e : A \to C$, $f : B \to D$, $g_n : A \to B$, $h_n : C \to D$, $n \in \omega$, in* $\mathbf{Pos}\,(\dot{\omega})$ *(resp.* $\mathbf{Pos}\,(\omega)$*) such that e is a dense (resp. strongly dense) map, $\{g_n\}$ is a chain having l.u.b. $g = \bigsqcup g_n$, and the squares*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g_n"'] & C \arrow[d, "h_n"] \\ B \arrow[r, "f"'] & D \end{tikzcd}$$

are commutative for each $n \in \omega$, then there exists a map $h : C \to D$ such that $fg = he$, and $h = \bigsqcup h_n$.

Proof. By 2.3, $\dot\omega[e,D]$ is a p.c. monomorphism. As $\{h_n e\}=\{fg_n\}$ is a chain with l.u.b. fg in $\dot\omega[A,D]$, $\{h_n\}$ is also a chain and has a l.u.b. $h=\bigsqcup h_n$. But $he=\bigsqcup h_n e=\bigsqcup fg_n=fg$. The proof for the case **Pos** (ω) is completely similar. $\square$

2.5. Notation. In the rest of the paper, several results will be proved jointly for the three cases where the base category **B** is either **Pos** or **Pos** $(\dot\omega)$ or **Pos** (ω). We will find sometimes convenient to reason on a generic base category **B**, which can be any of the above. Similarly $\mathbf{B}\hookrightarrow\mathbf{B}'$ will be generic notation for the three possible inclusions: **Pos** $(\omega)\hookrightarrow$**Pos**, **Pos** $(\dot\omega)\hookrightarrow$**Pos**, **Pos** $(\omega)\hookrightarrow$**Pos** $(\dot\omega)$ and $C:\mathbf{B}'\to\mathbf{B}$ will denote the corresponding left adjoint or completion, i.e. $C=\wedge,\ \dot\wedge,\ \sim$. The prefix **B** will sometimes be used to generically qualify an entity. For instance a **B**-algebra will mean an algebra with underlying carrier and operations in **B**, and a **B**-preorder on A will be a preorder containing $\geq A$ for $\mathbf{B}=\mathbf{Pos}$, and a $\dot\omega$(resp. ω)-continuous preorder on A for $\mathbf{B}=\mathbf{Pos}$ $(\dot\omega)$ (resp. **Pos** (ω)). The internal homs in **B** will be denoted $\mathbf{B}[A,B]$, A, $B\in\mathbf{B}$.

3. Classical varieties of chain-complete algebras

In this section, we present basic properties and constructions for Σ-algebras, and classical varieties of Σ-algebras, on the three categories **Pos**, **Pos** $(\dot\omega)$ and **Pos** (ω). Along the way, we get the algebraicity (stronger than monadicity) of the forgetful functors. As the three base categories are locally presentable, several of the results could be obtained from the general theory for those categories [42]. We have taken, however, a more direct approach, which has the advantage of allowing a straightforward generalization to categories of the form **Pos** (Z), for Z a crossed-down subset system [7, 82], which are by no means locally presentable. For simplicity of exposition, we shall develop the case of algebras on one underlying poset, and leave as an exercise the easy, but useful, generalization to "heterogeneous" or "many-sorted" algebras, i.e., algebras with a family of underlying posets. The classical varieties of heterogeneous algebras on the base categories $\mathbf{B}=\mathbf{Pos}$, **Pos** $(\dot\omega)$, **Pos** (ω), can be described as categories of product-preserving functors from a Bénabou [14] algebraic theory to **B**, just in the same way as product-preserving functors from a Lawvere [61] theory describe classical varieties in the one-sorted case.

For Σ a ranked alphabet, i.e. a map $\#:\Sigma\to\omega$, a Σ-algebra structure on $A\in\mathbf{B}$ is as usual the giving of operations $A\sigma:A^{\#\sigma}\to A$, which are **B**-morphisms. Given **B**-Σ-algebras A and B as above, a **B**-morphism $f:A\to B$ is a Σ-homomorphism iff $B\sigma\cdot f^{\#\sigma}=A\sigma.\,f$, for each $\sigma\in\Sigma$. We shall denote $\mathbf{B}_\Sigma$ the category of **B**-Σ-algebras and Σ-homomorphisms, and by $U_\Sigma:\mathbf{B}_\Sigma\to\mathbf{B}$ the obvious forgetful functor.

As **B** is complete, $\mathbf{B}_\Sigma$ is also complete, because U_Σ *creates limits* as it is well-known. A map $m:A\to B$ in $\mathbf{B}_\Sigma$ is mono iff $U_\Sigma m$ is mono in **B**. Hence $\mathbf{B}_\Sigma$ is well-powered. As U_Σ is faithful, $U_\Sigma e$ epi in **B** forces e epi in $\mathbf{B}_\Sigma$. Hence any factorization system $\mathcal{E}$, $\mathcal{M}$ in **B** defines a class $\mathcal{E}_\Sigma$ of epis and a class $\mathcal{M}_\Sigma$ of monos in $\mathbf{B}_\Sigma$ by taking: $e\in\mathcal{E}_\Sigma\Leftrightarrow U_\Sigma e\in\mathcal{E}$, $m\in\mathcal{M}_\Sigma\Leftrightarrow U_\Sigma m\in\mathcal{M}$.

3.1. Lemma (Homomorphism Theorem). *For* $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$ *and* $\mathcal{E}$, $\mathcal{M}$ *= coequalizer-mono and epi-full mono in* **Pos**; *extremal epi-mono, strongly dense-full mono, epi-extremal mono in* $\mathbf{Pos}(\dot{\omega})$ *and* $\mathbf{Pos}(\omega)$; *dense-p.c. mono in* $\mathbf{Pos}(\dot{\omega})$, *the pair* $\mathcal{E}_\Sigma$, $\mathcal{M}_\Sigma$ *gives a factorization system in* $\mathbf{B}_\Sigma$.

Proof. For $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, by 2.2 and the diagonal lemma for factorization systems [54], given a map $f : A \to B$ in $\mathbf{B}_\Sigma$, if $f = A \xrightarrow{e} \operatorname{Im} f \xrightarrow{m} B$ is its $\mathcal{E}$-$\mathcal{M}$ factorization in $\mathbf{B}$, there is a unique Σ-algebra structure on $\operatorname{Im} f$ making e and m Σ-homomorphisms, i.e. $e \in \mathcal{E}_\Sigma$, $m \in \mathcal{M}_\Sigma$. The remaining details, as well as the case $\mathbf{B} = \mathbf{Pos}$, are immediate. $\square$

Let $A \in \mathbf{B}_\Sigma$. A $\mathbf{B}$-preorder R on (the underlying poset of) A is said *congruent* for A iff

$$\forall \sigma \in \Sigma, \text{ if } a_1 \mathrel{R} a'_1, \dots, a_{\#\sigma} \mathrel{R} a'_{\#\sigma}, \text{ then}$$
$$A\sigma(a_1, \dots, a_{\#\sigma}) \mathrel{R} A\sigma(a'_1, \dots, a'_{\#\sigma}).$$

Note that if $f : A \to B$ is a map in $\mathbf{B}_\Sigma$, R_f is a congruent B-preorder on A.

3.2. Lemma. *For* $\mathbf{B} = Pos$, *resp.* $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, *given* $A \in \mathbf{B}_\Sigma$ *and* R *a congruent* $\mathbf{B}$*-preorder on* A, *there exists a unique* Σ*-algebra structure on* A/R, *resp.* $(\widetilde{A/R})$ *making* q_R, *resp.* $\tilde{q}_R$ *into a* Σ*-homomorphism. In addition, given* $f : A \to B$ *in* $\mathbf{B}_\Sigma$, *there exists a* $\mathbf{B}_\Sigma$*-map* f *such that* $\bar{f} \cdot q_R = f$, *resp.* $\bar{f} \cdot \tilde{q}_R = f$ *iff* $R_f \supseteq R$, *and such an* $\bar{f}$ *is then unique.*

Proof. (Cf. [31].) For $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, note that to require R $\mathbf{B}$-congruent on A is equivalent to impose that the $\mathbf{B}$-preorder R satisfies $(A\sigma^2)^{-1}(R) \supseteq R^{\#\sigma}$, where, by definition $(a_1, \dots, a_{\#\sigma}) R^{\#\sigma} (a'_1, \dots, a'_{\#\sigma})$ iff $a_i \mathrel{R} a'_i$, $i = 1, \dots, \#\sigma$. But $R^{\#\sigma}$ is the $\mathbf{B}$-preorder induced by $q_R^{\#\sigma}$. Hence, for any $\sigma \in \Sigma$ there exists a unique map $(\widetilde{A/R})\sigma : (\widetilde{A/R})^{\#\sigma} \to (\widetilde{A/R})$ with $(\widetilde{A/R})\sigma \cdot \tilde{q}_R^{\#\sigma} = \tilde{q}_R \cdot A\sigma$, because, by 2.2, $\tilde{q}_R^{\#\sigma}$ is a nice epimorphism. This proves the first assertion. For f as in the second assertion, we only are to see that the corresponding induced $\mathbf{B}$-map $\bar{f}$ is a Σ-homomorphism. This follows from $\tilde{q}_R^{\#\sigma}$ epi $\forall \sigma \in \Sigma$, by an easy diagram chase. For $\mathbf{B} = \mathbf{Pos}$ do the same, or see [19]. $\square$

From Section 2 we know that a full subposet $E \subseteq A^2$ is a (canonical) kernel-pair on $A \in \mathbf{B}$ iff (the underlying set of) E is of the form E_R for some $\mathbf{B}$-preorder R on A. Given an algebra $A \in \mathbf{B}_\Sigma$ we say that a canonical kernel-pair E on its underlying poset is a *congruence* on A iff E is a subalgebra of A^2.

3.3. Lemma (U_Σ Creates coequalizers of congruences). *For* $\mathbf{B} = \mathbf{Pos}$, *resp.* $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, $A \in \mathbf{B}_\Sigma$, *and* E *a congruence on* A *there is a unique* Σ*-algebra structure on* $A/\bar{E}$, *resp.* $(\widetilde{A/\bar{E}})$ ($\bar{E}$ *the* $\mathbf{B}$*-preorder generated by* E) *making* $q_{\bar{E}}$, *resp.* $\tilde{q}_{\bar{E}}$, *a* Σ*-homomorphism. In addition,* $q_{\bar{E}}$, *resp.* $\tilde{q}_{\bar{E}}$, *is also the coequalizer of its kernel-pair* E *in the category* $\mathbf{B}_\Sigma$.

Proof. For $\mathbf{B}=\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, E with projections $\pi_1,\pi_2 : E\to A$ is kernel-pair of $\tilde{q}_E$ in $\mathbf{B}$. Hence, by interchangeability of limits, $E^{\#\sigma}$, $\pi_1^{\#\sigma}$, $\pi_2^{\#\sigma}$, is kernel-pair of $\tilde{q}_E^{\#\sigma}$ for each $\sigma\in\Sigma$. But $q_E^{\#\sigma}$ is coequalizer by 2.2, hence coequalizer of $\pi_1^{\#\sigma}$, $\pi_2^{\#\sigma}$. Use now the fact that π_1, π_2 are Σ-homomorphisms by hypothesis, and the universal property of $\tilde{q}_E^{\#\sigma}$, to get the unique maps $(\widetilde{A/\bar{E}})\sigma : (\widetilde{A/\bar{E}})^{\#\sigma}\to(\widetilde{A/\bar{E}})$, making $\tilde{q}_E$ Σ-homomorphism. To see $\tilde{q}_E$ coequalizer of π_1, π_2 in $\mathbf{B}_\Sigma$ chase the corresponding diagram or remark the (just proved) fact that $\bar{E}$ is congruent and use 3.2. The case $\mathbf{B}=\mathbf{Pos}$ follows similarly, noticing that $\mathbf{Pos}$ is cartesian closed, and coequalizers in $\mathbf{Pos}$ are closed under composition. □

3.4. Remark. Note that for $A\in\mathbf{B}_\Sigma$, the sets of congruent B-preorders on A and of congruences on A form both complete lattices, and that the maps $R\mapsto E_R$, $E\mapsto\bar{E}$, relating $\mathbf{B}$-preorders and kernel-pairs on the underlying poset of A restrict, by 3.3, to give a galois connection between congruent B-preorders and congruences on A. Given homomorphisms $f,g : A\to B$ in $\mathbf{B}$, their coequalizer is the coequalizer of the smallest congruence containing the pairs (fa,ga), (ga,fa), $\forall a\in A$. More generally, the "quotient" of an algebra $A\in\mathbf{B}_\Sigma$ by imposing extra order relations in a given set of pairs $Q\subseteq A^2$ can be constructed, by generating the smallest congruent $\mathbf{B}$-preorder containing Q, and has the obvious universal property.

The construction of free Σ-algebras can be done in any cartesian closed category with countable coproducts by just imitating the word algebra construction of classical Universal Algebra [70, 88]. The intuitive observation, which hints the formula, is as follows: if $W_\Sigma(A)$ denotes the word algebra on the set A, the unique map $! : A\to 1$ induces $W_\Sigma(!) : W_\Sigma(A)\to W_\Sigma(1)$, which is just substitution of any appearance of elements of A in a word by 1. If we denote $\#(w)\in\omega$ the number of appearances of 1 in a word $w\in W_\Sigma(1)$, it is then clear that we have a bijection $A^{\# w}\simeq W_\Sigma(!)^{-1}\{w\}$ for any $w\in W_\Sigma(1)$. Hence the formula

$$W_\Sigma(A)\simeq\coprod_{w\in W_\Sigma(1)} A^{\# w},$$

whose right hand side gives the construction of the free Σ-algebra $F_\Sigma(A)$ on an object A, for an arbitrary cartesian closed category with countable coproducts. The operations are then defined as adequate induced maps from coproducts. For $\mathbf{B}=\mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, as products and coproducts are set-like, the formula boils down to say that for $A\in\mathbf{B}$, $F_\Sigma(A)$ has underlying set the ordinary word algebra on the underlying set of A, and its order is only defined on words w, w' such that $F_\Sigma(!)w=F_\Sigma(!)w'$, and is then given by the componentwise ordering of A^n, $n=\#F_\Sigma(!)(w)$.

For $\mathbf{B}=\mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, $\mathbf{B}_\Sigma$ is then cocomplete: 3.4 gives coequalizers, and coproducts $\oplus A_i$ are then obtained by the well-known method (see for instance [66, 59]) of forming a coequalizer from $F_\Sigma(\coprod U_\Sigma A_i)$ under, say, the smallest congruence containing all the congruences $E\varepsilon_i$ induced by the counit homomorphisms $\varepsilon_i : F_\Sigma U_\Sigma(A_i)\to A_i$.

3.5. Lemma. *For* $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, $U_\Sigma : \mathbf{B}_\Sigma \to \mathbf{B}$ *creates filtered colimits.*

Proof. Adapting the proof for sets (see [70]) one sees that in a cartesian closed category finite products commute with filtered colimits. If $\{A_i\}$ is a filtered diagram in $\mathbf{B}_\Sigma$, the universal property of (colim $U_\Sigma A_i)^{\#\sigma}$ provides the unique operations making the injections homomorphisms. The rest is easy exercise. □

Before proceeding to consider classical varieties, we shall make a few remarks on the relationship of above results to axiomatic theories of monadic and algebraic functors [61, 66, 37, 87, 72]. We have seen that $U_\Sigma : \mathbf{B}_\Sigma \to \mathbf{B}$

(i) creates limits,
(ii) creates coequalizers of congruences,
(iii) has a left adjoint,
(iv) creates filtered colimits.

A functor $U : \mathbf{K} \to \mathbf{B}$ is *algebraic* in the sense of Pfender [87], if it satisfies (i), (ii), (iii). This is strictly stronger than monadic. In fact, [87], for $\mathbf{B}$ finitely complete, U is algebraic iff it is monadic and satisfies (ii). In the category of sets both concepts are coextensive [61, 66, 72], but in general categories, nicer properties hold for algebraic than for monadic functors. For instance, the composition of algebraic functors is algebraic, and if $\mathbf{B}$ complete, well-powered and cocomplete, $\mathbf{K}$ is also so. Both properties need not hold for monadic functors [53, 1]. For other properties see [87, 88, 59]. Note finally that monadicity of U_Σ above can be easily checked directly, for instance by Pare's criterion.

For $B = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, a *classical variety* of algebras on $\mathbf{B}$, presented by operations Σ and equations Γ, is defined in the usual way: it is the full subcategory of $\mathbf{B}_\Sigma$ formed by those algebras which satisfy the axioms in Γ. If T_Σ denotes the free (Lawvere) algebraic theory on Σ, and $T = T_{\Sigma/\Gamma}$ the quotient theory under the equations Γ, the variety of Σ,Γ-algebras can be described in a functorial, presentation-independent way (cf. [61]) as the category with objects (canonical) product-preserving functors $A : T \to \mathbf{B}$, and morphisms natural transformations. We then denote $\mathbf{B}_T$ such a variety. The following properties are easy to check, chasing the adequate diagrams:

3.6. Proposition. *For* $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, *let* $T = T_{\Sigma/\Gamma}$, *and* $\mathbf{B}_T \hookrightarrow \mathbf{B}_\Sigma$ *the corresponding full inclusion. Then* $\mathbf{B}_T$ *is closed in* $\mathbf{B}$ *under:*

(i) *products,*
(ii) *monomorphisms,*
(iii) $\mathscr{E}_\Sigma$*-quotients, for* ε *= epimorphisms in* $\mathbf{B}$,
(iv) *filtered colimits.*

3.7. Remark. (i) and (ii) are enough to show that $\mathbf{B}_T \hookrightarrow \mathbf{B}_\Sigma$ is extremal-epi-reflective subcategory [54], and (iii) together with 3.3 and 3.4 show that it is closed under coequalizers. It is then immediate to see that both the inclusion functor $\mathbf{B}_T \hookrightarrow \mathbf{B}_\Sigma$, and the forgetful functor $U_T := \mathbf{B}_T \hookrightarrow \mathbf{B}_\Sigma \xrightarrow{U_\Sigma} \mathbf{B}$ are algebraic, and create filtered colimits. For any $A \in \mathbf{B}_\Sigma$ the reflection map $A \to R_T A$ is the expected one: the coequalizer under the smallest congruence generated by all the pairs $(\alpha(a_1, \dots, a_n), \beta(a_1, \dots, a_n))$, $(\beta(a_1, \dots, a_n), \alpha(a_1, \dots, a_n))$, $(a_1, \dots, a_n) \in A^n$, for all equations $(\alpha, \beta) \in \Gamma$, of all arities $n \in \omega$, in a given presentation Σ, Γ of T.

3.8. Corollary. *For* $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot\omega)$, $\mathbf{Pos}(\omega)$, $T = T_{\Sigma/\Gamma}$,

(a) $\mathbf{B}_T$ *is complete and cocomplete, the colimits in* $\mathbf{B}_T$ *being the reflection under* R_T *of the ones computed in* $\mathbf{B}_\Sigma$. $\mathbf{B}_T$ *is also locally presentable* (cf. [42], 11.3).

(b) *By* 3.6(iii), 3.1, 3.2 *and* 3.3 *hold replacing* Σ *by* T.

(c) *For* $H : T \to T'$ *a theory morphism, the induced functor*

$$\mathbf{B}_H : \mathbf{B}_{T'} \to \mathbf{B}_T : (T' \xrightarrow{A} B) \mapsto (T \xrightarrow{H} T' \xrightarrow{A} \mathbf{B})$$

is algebraic and creates filtered colimits.

(This follows from 3.5, 3.6(iv) and the triangle theorem for algebraic functors [88, 59], applied to the triangle $U_{T'} = U_T \bullet \mathbf{B}_H$. The left adjoint F_H can be computed as in [72], 3.1.29 or, more classically, as the coequalizer of $F_{T'} U_T A$ (where $F_{T'}$ is the left adjoint of $U_{T'}$) under the congruence generated by the pairs (hu, hv), for all (u, v) in the congruence induced by the counit $\varepsilon_T A : F_T U_T A \to A$, and h the map:

$$F_T U_T A \xrightarrow{F_T \eta_{T'} U_T A} F_T U_T F_{T'} U_T A \xrightarrow{T\mathbf{B}_H F_{T'} U_T A} \mathbf{B}_H F_{T'} U_T A).). \quad \Box$$

We shall now make a remark on completions of algebras and give a lemma providing a universal construction to impose limits in an algebra $A \in \mathbf{Pos}(\dot\omega)_T$.

3.9. Remark. For $\mathbf{B} \hookrightarrow \mathbf{B}'$ one of the three possible inclusions between $\mathbf{Pos}$, $\mathbf{Pos}(\dot\omega)$ and $\mathbf{Pos}(\omega)$, and T an algebraic theory, we have a corresponding inclusion at the level of algebras: $(T \xrightarrow{A} \mathbf{B}) \mapsto (T \xrightarrow{A} \mathbf{B} \hookrightarrow \mathbf{B}')$, such that

$$\begin{tikzcd}
\mathbf{B}_T \arrow[r] \arrow[d, "U_T"'] & \mathbf{B}'_T \arrow[d, "U'_T"] \\
\mathbf{B} \arrow[r] & \mathbf{B}'
\end{tikzcd}$$

Now, if $C : \mathbf{B}' \to \mathbf{B}$ denotes the corresponding completion, as C is product-preserving (see Section 2), the left adjoint $\bar{C}$ for the inclusion $\mathbf{B}_T \hookrightarrow \mathbf{B}'_T$ can be easily described as follows:

$$\bar{C} : (T \xrightarrow{A} \mathbf{B}') \mapsto (T \xrightarrow{A} \mathbf{B}' \xrightarrow{C} \mathbf{B}),$$

and has unit maps

$$\begin{tikzcd}
T \arrow[r, "A"] & \mathbf{B}' \arrow[r, "\mathrm{id}"] \arrow[dr, "C"'] & \mathbf{B}' \arrow[d, phantom, "\Downarrow\eta"] \\
& & \mathbf{B} \arrow[u]
\end{tikzcd}$$

So we have the diagram

$$\begin{tikzcd} \mathbf{B}'_T \arrow[r, "\bar{C}"] \arrow[d, "U'_T"'] & \mathbf{B}_T \arrow[d, "U_T"] \\ \mathbf{B}' \arrow[r, "C"'] & \mathbf{B} \end{tikzcd}$$

By construction, the unit maps for C are full monomorphisms in the three cases, and also dense for the case $\mathbf{Pos}(\omega) \hookrightarrow \mathbf{Pos}(\dot{\omega})$. Note that completely similar remarks apply, for $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, to the forgetful functor $V: \mathbf{B} \to \mathbf{Set}$, because its left adjoint (discrete structure) $D: \mathbf{Set} \to \mathbf{B}$ is product-preserving. Hence we have a corresponding $\bar{D}: \mathbf{Set}_T \to \mathbf{B}_T$, left adjoint for $\bar{V}: \mathbf{B}_T \to \mathbf{Set}_T : A \mapsto VA$, which sends an ordinary algebra A to its discrete poset structure DA.

3.10. Lemma. *Let T be an algebraic theory, $A \in \mathbf{Pos}(\dot{\omega})_T$, and $\mathcal{F} = \{\{ai_n\}\}_{i \in I}$ a family of chains in A. There exists an algebra $A_{\mathcal{F}} \in \mathbf{Pos}(\dot{\omega})_T$ and a dense homomorphism $\eta_{\mathcal{F}}: A \to A_{\mathcal{F}}$ such that $\bigsqcup_n \eta_{\mathcal{F}} a^i_n$ exists for any chain $\{a^i_n\}$, and given a continuous homomorphism $f: A \to B$ to an algebra $B \in \mathbf{Pos}(\dot{\omega})_T$ such that $\bigsqcup_n fa^in$ exists for any chain $\{a^i_n\} \in \mathcal{F}$, there exists a unique continuous homomorphism $\bar{f}: A_{\mathcal{F}} \to B$ such that $\bar{f} \cdot \eta_{\mathcal{F}} = f$.*

Proof. Consider the unit map $\tilde{\eta}A : A \to \tilde{A}$, corresponding to the completion determined by the inclusion $\mathbf{Pos}(\omega)_T \hookrightarrow \mathbf{Pos}(\dot{\omega})_T$. Notice that in $\mathbf{Pos}(\dot{\omega})_T$, the full subalgebras of an algebra are closed under intersection and form a complete lattice. Let $A_{\mathcal{F}}$ be the intersection of all full subalgebras of $\tilde{A}$, in $\mathbf{Pos}(\dot{\omega})_T$, which contain $\tilde{\eta}A(A)$ and $\bigsqcup_n \tilde{\eta}Aa^i_n$ for each $\{a^i_n\} \in \mathcal{F}$, and $\eta_{\mathcal{F}}: A \to A_{\mathcal{F}}$ the correstriction of $\tilde{\eta}A$ to $A_{\mathcal{F}}$. $\eta_{\mathcal{F}}$ has the desired universal property because, given $f: A \to B$ as above, and considering the diagram:

$$\begin{tikzcd} A \arrow[rr, "\tilde{\eta}A"] \arrow[dr, "\eta_{\mathcal{F}}"] \arrow[dd, "f"'] & & \tilde{A} \arrow[dd, "\tilde{f}"] \\ & A_{\mathcal{F}} \arrow[ur, "ik"'] \arrow[dl, dashed, "\bar{f}"] & \\ B \arrow[rr, "\tilde{\eta}B"'] & & \tilde{B} \end{tikzcd}$$

there is a *function* $\bar{f}$ such that $\tilde{f}k = \tilde{\eta}B \cdot \bar{f}$, due to the fact that $\tilde{\eta}B$ is full monomorphism and $\tilde{f}(A_{\mathcal{F}}) \subseteq \tilde{\eta}B(B)$, because $\tilde{\eta}B(B)$ is full subalgebra of $\tilde{B}$ in $\mathbf{Pos}(\dot{\omega})_T$, $\tilde{f}(A_{\mathcal{F}})$ is contained in the intersection of the family of full subalgebras of $\tilde{B}$ containing $\tilde{f}\tilde{\eta}A(A)$ and $\bigsqcup_n \tilde{f}\tilde{\eta}A(a^i_n)$ for each $\{a^i_n\} \in \mathcal{F}$, and $\tilde{\eta}B(B)$ belongs to that family. $\bar{f}$ is a continuous homomorphism because (exercise) in $\mathbf{Pos}(\dot{\omega})_T$ full monomorphisms are *embeddings* [54] (see [82], 4.10.3). $\bar{f}$ is then unique, because $\eta_{\mathcal{F}}$ is dense, given that $\overline{\eta_{\mathcal{F}}(A)}$ is a full subalgebra of $\tilde{A}$ which contains $\tilde{\eta}A(A)$ and $\bigsqcup_n \tilde{\eta}Aa^i_n$ for each $\{a^i_n\} \in \mathcal{F}$. $\square$

We finish with a few comments about quotients. Notice that for $T = T_{\Sigma/\Gamma}$, and $A \in \mathbf{B}_T$, if $f: A \to B$ is epi in $\mathbf{B}$, there is at most one T-algebra structure on B making f a homomorphism. Hence $\mathbf{B}_T$ is $\mathscr{E}_T$ $(:= \mathscr{E}_\Sigma)$-cowellpowered, because $\mathbf{B}$ is $\mathscr{E}$-cowellpowered, for $\mathscr{E}$ in any factorization system of $\mathbf{B}$. The $\mathscr{E}_T$-quotients of an algebra A then form a complete lattice (easy, see [80]). Similarly, the nice quotients and the coequalizer quotients form complete lattices, anti-isomorphic to those of congruent $\mathbf{B}$-preorders and congruences respectively.

4. Varieties of chain-complete algebras

In this section we pass from classical varieties to, what seems to be (see 4.9), the right notion of variety in the three instances of $\mathbf{B}$. Varieties are now described by $\mathbf{B}$-theories, whose homsets are objects in $\mathbf{B}$. This allows inequalities $\alpha \geq \beta$ between operations and, in the cases $\mathbf{Pos}(\dot{\omega})$ and $\mathbf{Pos}(\omega)$, limit operations $\alpha = \bigsqcup \alpha_n$. So a "logic", quite richer than the classical equational logic, is around. This richness actually demands an essential use of the point of view of Lawvere [61], because in the cases $\mathbf{Pos}(\dot{\omega})$ and $\mathbf{Pos}(\omega)$, the quotient $\mathbf{B}$-theories are not describable anymore by congruences as in $\mathbf{Set}$ or by preorders as in $\mathbf{Pos}$ [80]. Actually, without that viewpoint, the proper notion of variety would be, in our opinion, elusive, if not impossible to capture. Even though several of the results here are particular instance of the general approach to relative Universal Algebra in closed categories by Gray [49], Sols [98] and Borceaux [21], we have found necessary to develop special constructions, and to exploit the particular features of the categories at hand in order to prove the Birkhoff Theorem, which characterizes the classes of algebras which are varieties. We first show that the category of algebras of a $\mathbf{B}$-theory has all the expected good properties, then look at the structure-semantics adjointness, which yields the completeness theorem, and finally prove the Birkhoff Theorem. The case $\mathbf{B} = \mathbf{Pos}$ has been studied by Bloom [19], but we include it too for the sake of completeness.

As pointed out by Bénabou [14], for any indexing set I, the category $\mathbf{Th}_I$ of I-sorted, heterogeneous, algebraic theories is itself a category of $I^* \times I$-sorted algebras (I^* the free monoid on I) for the "algebraic theory of I-sorted algebraic theories". In particular, the category $\mathbf{Th}$ of ordinary, 1-sorted, (Lawvere) algebraic theories is a category of $1^* = \omega$-sorted algebras for the (ω-sorted) theory of 1-sorted theories. We can consider for $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, the category $\mathbf{BTh}$ of algebras in $\mathbf{B}$ for such a theory of theories. We then get the concepts of ordered, resp. $\dot{\omega}$-complete, resp. ω-complete, algebraic theory and monotonic, resp. continuous, theory morphism. In other words, $\mathbf{B}$-theories are ordinary theories with a $\mathbf{B}$-category [34] structure which makes the products $\mathbf{B}$-products, and morphisms are theory morphisms which are $\mathbf{B}$-functors. By definition, $\mathbf{BTh}$ is a (classical) variety of ω-sorted algebras on $\mathbf{B}$, with forgetful functor $U: \mathbf{BTh} \to \mathbf{B}^\omega$. Hence all the results of Section 3 apply, in their ω-sorted version, to $\mathbf{BTh}$. In particular, by 3.9, for $\mathbf{B} \hookrightarrow \mathbf{B}'$

any of the three possible inclusions we have a left adjoint $\bar{C}$ for the inclusion $\mathbf{BTh} \hookrightarrow \mathbf{B'Th}$ which satisfies $U\bar{C} = C^{\omega}U'$, and a left adjoint $\bar{D}$ for the forgetful functor $\bar{V}: \mathbf{BTh} \to \mathbf{Th}$, such that $U\bar{D} = D^{\omega}U''$. Note that any $\mathbf{B}$-theory T is a bijective quotient of a discrete theory by means of the counit map $\varepsilon T: \bar{D}\bar{V}T \to T$.

$\mathbf{B}$-theories define varieties of algebras on $\mathbf{B}$ just as ordinary theories define classical varieties. If T is a $\mathbf{B}$-theory, a *T-algebra* is a product ($=\mathbf{B}$-product) preserving $\mathbf{B}$-functor $A : T \to B$. A homomorphism of $\mathbf{B}$-algebras is a natural ($=\mathbf{B}$-natural, because 1 is a generator in $\mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$ and $\mathbf{Pos}(\omega)$) transformation. Hence we have a category, that we shall denote $\mathbf{B}_T$, and a forgetful functor $U_T: \mathbf{B}_T \to \mathbf{B}$. For an ordinary theory T, the category of algebras for the discrete $\mathbf{B}$-theory $\bar{D}T$ coincides with the classical variety $\mathbf{B}_T$ i.e. $\mathbf{B}_T = \mathbf{B}_{\bar{D}T}$. For completions of theories one has a similar result. First notice that for $\mathbf{B} \hookrightarrow \mathbf{B'}$ any of the three possible inclusions between $\mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$ and $\mathbf{Pos}(\omega)$, $\mathbf{B}$ is a $\mathbf{B'}$-category. Hence for T a $\mathbf{B'}$-theory we can also denote by $\mathbf{B}_T$ (by abuse of language) the category of product-preserving $\mathbf{B'}$-functors from T to $\mathbf{B}$, and natural transformations. This does not add anything new, because

4.1. Lemma. *Let $\mathbf{B} \hookrightarrow \mathbf{B'}$ be any of the three possible inclusions between* $\mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, *and* $\mathbf{Pos}(\omega)$, *T a $\mathbf{B'}$-theory and $\eta: T \to \bar{C}T$ the universal map for T. Then the functor $(-\bullet\eta): \mathbf{B}_{\bar{C}T} \to \mathbf{B}_T: A \mapsto A\bullet\eta$ (which clearly satisfies $U_T\bullet(-\bullet\eta) = U_{\bar{C}T}$) is an isomorphism of categories.*

Proof. For any $\mathbf{B}$-category $\mathbf{K}$, each $\mathbf{B'}$-product-preserving $\mathbf{B'}$-functor $A : T \to \mathbf{K}$ has a "full image factorization" as $T \xrightarrow{A_0} T_A^0 \xrightarrow{A^0} \mathbf{K}$ with A_0 in $\mathbf{B'Th}$ and $T_A^0 \in \mathbf{BTh}$, by taking $T_A^0(n,1) := \mathbf{K}[A_n, A_1]$. Taking $\mathbf{K} = \mathbf{B}$ and using the universal property of η one then sees that $(-\bullet\eta)$ is bijective on the objects. Taking $\mathbf{K} = \mathbf{B}^{\mathbf{2}}$, for $\mathbf{2} := \{0 \to 1\}$, and using again the universal property of η one gets $(-\bullet\eta)$ bijective on the morphisms. □

As in 3.9, we have a notion of completion of algebras which can be carried out at the level of the base categories.

4.2. Lemma. *Let $\mathbf{B} \hookrightarrow \mathbf{B'}$ be any of the three possible inclusions between* $\mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$ *and* $\mathbf{Pos}(\omega)$ *and T a $\mathbf{B'}$-theory. Then the inclusion $\mathbf{B}_T \hookrightarrow \mathbf{B'}_T$ has a $\mathbf{B'}$-left adjoint $\bar{C}$ defined by: $(T \xrightarrow{A} \mathbf{B'}) \mapsto (T \xrightarrow{A} \mathbf{B'} \xrightarrow{C} \mathbf{B})$, C the left adjoint for $\mathbf{B} \hookrightarrow \mathbf{B'}$. Hence $U_T\bullet\bar{C} = C\bullet U'_T$.*

Proof. The only difference with 3.9 is that now we have to use that C is not only product-preserving, but also a $\mathbf{B'}$-functor. This is immediate to see for the two cases with $\mathbf{B'} = \mathbf{Pos}$, and follows easily from 2.3 and 2.4 for the inclusion $\mathbf{Pos}(\omega) \hookrightarrow \mathbf{Pos}(\dot{\omega})$, using the fact that the maps $\tilde{\eta}A$, $A \in \mathbf{Pos}(\dot{\omega})$, are dense. Note finally that $\mathbf{B}_T$ and $\mathbf{B'}_T$, being full subcategories of $\mathbf{B'}$-functor categories, are $\mathbf{B'}$-categories, that their inclusion is a $\mathbf{B'}$-functor, and that $\bar{C}$ is $\mathbf{B'}$ functor because C is, hence $\mathbf{B'}$-left adjoint. □

We shall now consider the varieties defined by $\mathbf{B}$-theories. First notice that for $\mathbf{B}=\mathbf{Pos}$, $\mathbf{Pos}(\dot\omega)$, $\mathbf{Pos}(\omega)$ and $H: T' \to T$ a $\mathbf{B}$-theory morphism such that for each $n \in \omega$ it is $H_n: T'(n,1) \to T(n,1)$ an epimorphism in $\mathbf{B}$, the functor $\mathbf{B}_H: \mathbf{B}_T \to \mathbf{B}_{T'}: (T \xrightarrow{A} B) \mapsto (T' \xrightarrow{H} T \xrightarrow{A} \mathbf{B})$ is a *full embedding*. Because one can see $\mathbf{B}_H$ injective on the objects by taking the factorization $A=(T \xrightarrow{A_0} T_A^0 \xrightarrow{A^0} \mathbf{B})$ as in 4.1, and using H epimorphism in $\mathbf{BTh}$; one has $\mathbf{B}_H$ faithful because $U_T = U'_T \mathbf{B}_H$ and U_T, U'_T are faithful; and $\mathbf{B}_H$ is full because $f: AH \to BH$, T'-homomorphism (i.e. $\mathbf{B}$-natural transformation) is equivalent to the commutativity for each $n \in \omega$ of the outer diagram in

$$\begin{tikzcd}
T'(n,1) \arrow[d, "H_n"] \arrow[drr, "(AH)_n"] \arrow[dd, bend right, "(BH)_n"'] & & \\
T(n,1) \arrow[rr, "A_n"] \arrow[d, "B_n"] & & \mathbf{B}[A_n, A_1] \arrow[d, "\mathbf{B}[A_n, f_1]"] \\
\mathbf{B}[B_n, B_1] \arrow[rr, "\mathbf{B}(f_n, B_1)"'] & & \mathbf{B}[A_n, B_1]
\end{tikzcd}$$

which forces by Hn epi the commutativity of the square for each $n \in \omega$, i.e. $f: A \to B$ a T-homomorphism.

Given a $\mathbf{B}$-theory T, by a *presentation* of T we will mean a theory morphism $H: \bar{D}T_\Sigma \to T$ which is: surjective in the case $\mathbf{B}=\mathbf{Pos}$, dense in the case $\mathbf{B}=\mathbf{Pos}(\dot\omega)$, and strongly dense in the case $\mathbf{B}=\mathbf{Pos}(\omega)$. Note that any $\mathbf{B}$ theory has the bijective morphism $\varepsilon T: \bar{D}\bar{V}T \to T$, and as a consequence, any presentation H factors as $\bar{D}T_\Sigma \xrightarrow{\bar{D}\bar{V}H} \bar{D}\bar{V}T \xrightarrow{\varepsilon T} T$. By a *variety of algebras on* $\mathbf{B}$ we shall mean an embedding $\mathbf{B}_H: \mathbf{B}_T \to \mathbf{B}_\Sigma$ corresponding to a presentation $H: \bar{D}T_\Sigma \to T$ for some $T \in \mathbf{BTh}$. We shall say that $\mathbf{B}_T$ is *closed* under products, quotients, ... etc. in $\mathbf{B}_\Sigma$ (via $\mathbf{B}_H$) iff the image category $\mathbf{B}_H(\mathbf{B}_T)$ is closed in $\mathbf{B}_\Sigma$ under products, quotients, ... etc.

4.3. Proposition. *Let $\mathbf{B}_H: \mathbf{B}_T \to \mathbf{B}_\Sigma$ be a variety of Σ-algebras on $\mathbf{B}$. Then:*

(i) *For $\mathbf{B}=\mathbf{Pos}$, $\mathbf{B}_T$ is closed in $\mathbf{B}_\Sigma$ under products, full surjective quotients and filtered colimits.*

(ii) *For $\mathbf{B}=\mathbf{Pos}(\dot\omega)$, $\mathbf{B}_T$ is closed in $\mathbf{B}_\Sigma$ under products, persistently complete subalgebras, dense quotients and filtered colimits.*

(iii) *For $\mathbf{B}=\mathbf{Pos}(\omega)$, $\mathbf{B}_T$ is closed in $\mathbf{B}_\Sigma$ under products, full subalgebras, strongly dense quotients and filtered colimits.*

Proof. (i) has been proved in [19], except for the part of filtered colimits, which has a similar proof in the three cases. Suppose $\{A_i\}_{i \in I}$ is a filtered diagram in $\mathbf{B}_T$. As we have the factorization $H = \bar{D}T_\Sigma \to \bar{D}(H(T_\Sigma)) \hookrightarrow \bar{D}\bar{V}T \xrightarrow{\varepsilon T} T$, by 3.5 and 3.8(c) applied to $\bar{V}H: T_\Sigma \to \bar{V}T$, for any $\alpha: n \to 1$ in T, $n \in \omega$, we have in $\mathbf{B}$ a diagram of the form:

$$\begin{tikzcd}
A_i^n \arrow[r, "j_i"] \arrow[d, "A_i\alpha"'] \arrow[rr, bend left, "k_i^n"] & \coprod A_i \arrow[r, "q"] \arrow[d, "\coprod A_i\alpha"] & (\operatorname{colim} A_i)^n \arrow[d, "(\operatorname{colim} A_i)\alpha"] \\
A_i \arrow[r, "j'_i"] \arrow[rr, bend right, "k_i"'] & \coprod A_i \arrow[r, "q'"] & \operatorname{colim} A_i
\end{tikzcd}$$

with j_i, j'_i, k_i the corresponding injections and q, q' coequalizers in $\mathbf{B}$. For $\mathbf{B}=\mathbf{Pos}$, if $\alpha,\beta : n\to 1$ in $\bar{V}T=H(T_\Sigma)$ and $\alpha\geq\beta$ in T, then $\coprod A_i\alpha\geq\coprod A_i\beta$ and, by q surjective, $(\operatorname{colim} A_i)\alpha\geq(\operatorname{colim} A_i)\beta$. Hence $\operatorname{colim} A_i$ is in $\mathbf{B}_T$. For $\mathbf{B}=\mathbf{Pos}(\dot{\omega})$ ($\mathbf{Pos}(\omega)$), we are to see that $\operatorname{colim} A_i : \bar{V}T\to\mathbf{B}$ is actually a $\mathbf{B}$-functor $\operatorname{colim} A_i : T\to\mathbf{B}$. But given $\alpha,\alpha_m : n\to 1$, $m\in\omega$, in T such that $\alpha=\bigsqcup\alpha_m$ we have $\coprod A_i\alpha=\bigsqcup_m(\coprod A_i\alpha_m)$, and, by 2.4 and q coequalizer, $(\operatorname{colim} A_i)\alpha=\bigsqcup_m(\operatorname{colim} A_i)\alpha_m$, as wanted.

Closure under products is immediate, because U_Σ, $U_{\bar{V}T}$ and $\mathbf{B}_{\bar{V}H}$ create products by 3.7 and 3.8, and the functor $\prod A_i : \bar{V}T\to\mathbf{B}$ is a $\mathbf{B}$-functor because $(\prod A_i)\alpha=\prod A_i n \xrightarrow{\prod A_i\alpha} \prod A_i 1$, for any $\alpha : n\to 1$ in T.

For $\mathbf{B}=\mathbf{Pos}(\dot{\omega})$, if $A\in\mathbf{Pos}(\dot{\omega})_T$ and $k : A'\to A$ is a persistently complete Σ-subalgebra, k is also a $H(T_\Sigma)$-subalgebra by 3.6. But by hypothesis for each $n\in\omega$ it is

$$T(n,1)=\overline{HT_\Sigma(n,1)}=\bigcup_\beta(HT_\Sigma(n,1))_\beta.$$

Hence we shall prove that $k : A'\to A$ is a T-algebra if we extend stepwise the algebra $A' : H(T_\Sigma)\to\mathbf{Pos}(\dot{\omega})$ to a $\mathbf{Pos}(\dot{\omega})$-functor $A' : T\to\mathbf{Pos}(\dot{\omega})$ for which $k : A\to A'$ is natural. If $\alpha_m : n\to 1$, $m\in\omega$, is a chain in T with a limit $\alpha=\bigsqcup\alpha_m$, and the α_m are in $H(T_\Sigma)$, then for any $(a_1,\dots,a_n)\in A'^n$ we have, by k persistently closed, that

$$\begin{aligned}A\alpha(ka_1,\dots,ka_n)&=\bigsqcup A\alpha_m(ka_1,\dots,ka_n)=\bigsqcup kA'\alpha_m(a_1,\dots,a_n)\\&=k(\bigsqcup A'\alpha_m(a_1,\dots,a_n))\end{aligned}$$

and that we can define

$$A'\alpha(a_1,\dots,a_n):=\bigsqcup A'\alpha_m(a_1,\dots,a_n)=(\bigsqcup A'\alpha_m)(a_1,\dots,a_n)$$

(without conflict if α was already in $H(T_\Sigma)$) so that k is natural for any $\alpha\in(HT_\Sigma(n,1))_1$. The same definition of $A'\alpha$ extends A', making k natural, to $(HT_\Sigma(n,1))_{\beta+1}$ if we had it already defined on $(HT_\Sigma(n,1))_\beta$ and $\{\alpha_m\}\subseteq(HT_\Sigma(n,1))_\beta$ was a chain with limit α in $T(n,1)$. As for limit ordinals, β, it is $(HT_\Sigma(n,1))_\beta=\bigcup_{\gamma<\beta}(HT(n,1))_\gamma$, this gives, by transfinite induction, an extension $A : \bar{V}T\to\mathbf{Pos}(\dot{\omega})$, which is functorial, because A is and k^n is monomorphism $\forall n\in\omega$, and is a $\mathbf{Pos}(\dot{\omega})$-functor because if $\{\alpha_m\}\subseteq T(n,1)$ is a chain and $\alpha=\bigsqcup\alpha_m$, for any $(a_1,\dots,a_n)\in A'^n$ we have, by k persistently closed,

$$kA'\alpha(a_1,\dots,a_n)=\bigsqcup A\alpha_m(ka_1,\dots,ka_n)=k(\bigsqcup A'\alpha_m(a_1,\dots,a_n)),$$

which forces, by k injective, $A'\alpha=\bigsqcup A'\alpha_m$. Closure under full subalgebras for the case $\mathbf{B} = \mathbf{Pos}(\omega)$ follows from the above, by noticing that the full subobjects of an object $A \in \mathbf{Pos}(\omega)$ are exactly its p.c. subobjects in $\mathbf{Pos}(\dot{\omega})$.

For $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$ ($\mathbf{Pos}(\omega)$), $A \in \mathbf{B}_T$, and $e : A \to B$ a dense (strongly dense) Σ-quotient, we have a $H(T_\Sigma)$-quotient by 3.6(iii), and then one extends $B : H(T_\Sigma) \to \mathbf{B}$ to an algebra $B : \bar{V}T \to \mathbf{B}$ as in the case of subalgebras, by using 2.4 as induction step, and shows that it is a $\mathbf{B}$-functor by using 2.3, 2.4 and the fact that A is so. $\square$

4.4 Corollary. *For* $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$:

(a) *Any variety* $\mathbf{B}_T$ *of* Σ*-algebras on* $\mathbf{B}$*, being closed under products and subalgebras of the corresponding class, is* $\mathscr{E}_\Sigma$*-reflective* [54] *for* $\mathscr{E}$ *= surjectious, resp. dense maps, resp. strongly dense maps* (*i.e.* $\mathbf{B}_H$ *has a left adjoint* R_H *such that for each* $A \in \mathbf{B}_\Sigma$ *the unit* $\eta A : A \to R_H A \cdot H$*, is in* $\mathscr{E}_\Sigma$). *Hence* $\mathbf{B}_T$ *is complete and cocomplete, because* $\mathbf{B}_\Sigma$ *is, and* $\mathbf{B}_H$ *creates limits.* 3.3 *and* 3.4*, together with above proposition, show that* $\mathbf{B}_T$ *is closed under coequalizers in* $\mathbf{B}_\Sigma$*. It then follows that* $\mathbf{B}_H : \mathbf{B}_T \to \mathbf{B}_\Sigma$*, and* $U_T = U_\Sigma \cdot \mathbf{B}_H$ *are algebraic, and by above proposition and* 3.5*, create filtered colimits. This in turn implies,* [42], 10.3, [80]*, that* $\mathbf{B}_T$ *is* ω*-locally presentable for* $\mathbf{B} = \mathbf{Pos}$*, and* ω_1*-locally presentable for* $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$.

(b) *By above proposition,* 3.2 *and* 3.3 *hold, changing* Σ *by* $T \in \mathbf{BTh}$*, and* 3.1 *also holds in the same way, for all the factorization systems* $\mathscr{E}$, $\mathscr{M}$ *there with* $\mathscr{E} \subsetneq$ *epis, resp. dense maps, resp. strongly dense maps. For* $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$ *Lemma* 3.10 *holds replacing ordinary theories by* $\mathbf{Pos}(\dot{\omega})$*-theories, because the universal map* $\eta_{\mathscr{E}}$ *there is dense.*

(c) $\mathbf{B}_T$ *is a* $\mathbf{B}$ *category* (*see* 4.2) *and* U_T *is* $\mathbf{B}$*-monadic because* F_Σ *is* $\mathbf{B}$*-functor by construction, and* R_H *is* $\mathbf{B}$*-functor by surjectivity and naturality of the reflection maps in the case* $\mathbf{B} = \mathbf{Pos}$*, and by* $\mathscr{E}_\Sigma$*-reflectiveness, naturality and* 2.4 *for* $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$.

(d) *For any* $\mathbf{B}$*-theory morphism* $G : T \to T'$*, the functor* $\mathbf{B}_G : \mathbf{B}_{T'} \to \mathbf{B}_T$ *is algebraic and creates filtered colimits.* $\square$

4.5. Remark. The results on local presentability in 4.4(a) have a model-theoretic meaning which deserves a few words. They allow to connect the particularly nice "internal, finitary, equational" description, relative to $\mathbf{B}$, of the categories $\mathbf{B}_T$, by means of "equational" $\mathbf{B}$-theories T, to the syntactical descriptions which are possible at an absolute level. By 4.4(a), there is a Gabriel–Ulmer ω-theory (see [23] for a characterization of the kind of finitary, Horn-like, first-order logic involved) of which $\mathbf{Pos}_T$, $T \in \mathbf{PosTh}$, is the category of models in $\mathbf{Set}$, and a Gabriel–Ulmer ω_1-theory (corresponding to a special kind of infinitary first-order logic) of which $\mathbf{B}_T$, $T \in \mathbf{BTh}$ is the category of models in $\mathbf{Set}$, for $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$. Finally, something can be said about the description of categories $\mathbf{B}_T$, for $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, at the level of the category $\mathbf{Pos}$. The functor $W_T := \mathbf{B}_T \xrightarrow{U_T} \mathbf{B} \hookrightarrow \mathbf{Pos}$ will not be algebraic in general, because $\mathbf{B} \hookrightarrow \mathbf{Pos}$ is monadic but not algebraic, as can be seen by taking the kernel-pair of a non-surjective coequalizer q in $\mathbf{B}$ (see [80], 3.10) and the coequalizer-mono factorization of the map q in $\mathbf{Pos}$. However, there is an "internal, ω_1-equational" description of $\mathbf{B}_T$ relative to $\mathbf{Pos}$ given by

4.6. Proposition. *For* $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$ *and* $T \in \mathbf{BTh}$, *the functor* $W_T : \mathbf{B}_T \to \mathbf{Pos}$ *is* **Pos**-*monadic and creates* ω_1-*filtered colimits.*

Proof. W_T creates ω_1-filtered colimits because $\mathbf{B} \hookrightarrow \mathbf{Pos}$ does [80], 4.1, and U_T creates filtered colimits. W_T and its left adjoint $F_T \cdot C$ are clearly **Pos**-functors. To see that W_T is monadic, consider a pair $f, g : A \to B$ in $\mathbf{B}_T$ such that the pair f, g has an absolute coequalizer $q : B \to Q$ in **Pos**. As $\mathbf{B} \hookrightarrow \mathbf{Pos}$ is monadic, q is also a coequalizer of f, g in **B**. On the other hand, Cq remains an absolute coequalizer of $Cf, Cg : CA \to CB$ in **B**, and, by U_T monadic, there is a unique T-algebra structure on CQ making Cq a coequalizer in $\mathbf{B}_T$ of Cf, Cg (notice 3.9). Let now $\alpha : n \to 1$ be any operation in T. As q^n is an absolute coequalizer of f^n, g^n in **Pos**, there is a unique monotonic $Q\alpha : Q^n \to Q$ such that $Q\alpha \cdot q^n = B\alpha \cdot q$ thus making q a $\bar{V}T$-homomorphism in $\mathbf{Pos}_{\bar{V}T}$. If we show Q continuous, we will be done, because by 3.3, 3.4, 3.8(b) and 4.3, this will give a unique T-algebra structure on Q making q T-homomorphism and coequalizer of f, g in $\mathbf{B}_T$. Now, $\eta A : A \to CA$, $\eta B : B \to CB$, $\varepsilon A : CA \to A$, $\varepsilon B : CB \to B$ and $\eta Q : Q \to CQ$ are $\bar{V}T$-homomorphisms in $\mathbf{Pos}_{\bar{V}T}$ by 3.9. Reasoning on each operation $\alpha : n \to 1$ in T, and chasing the adequate prism in **Pos**, involving f, g, q, Cf, Cg, Cq, $A\alpha$, $B\alpha$, $Q\alpha$, ε and η, one gets, by the universal property of $(Cq)^n = C(q^n)$, absolute coequalizer of $(Cf)^n$, $(Cg)^n$ in **Pos**, that the map $\varepsilon Q : CQ \to Q$ is a $\bar{V}T$-homomorphism in $\mathbf{Pos}_{\bar{V}T}$. Now, notice that if $\{\vec{d}_m\} := \{(d_1^m, \ldots, d_n^m)\}$ is a chain in Q^n, and $\vec{d} = \bigsqcup \vec{d}_m$, it is (by the continuity of $\varepsilon(Q^n) = (\varepsilon Q)^n := \varepsilon^n$) $\vec{d} = \varepsilon^n(\bigsqcup_m \eta^n \vec{d}_m)$. Hence we have

$$\begin{aligned} Q\alpha(\vec{d}) &= Q\alpha \cdot \varepsilon^n\left(\bigsqcup_m \eta^n \vec{d}_m\right) = \varepsilon \cdot CQ\alpha\left(\bigsqcup_m \eta^n \vec{d}_m\right) \\ &= \bigsqcup_m \varepsilon \cdot CQ\alpha \cdot \eta^n(\vec{d}_m) = \bigsqcup_m Q\alpha \cdot \varepsilon^n \cdot \eta^n(\vec{d}_m) = \bigsqcup_m Q\alpha(\vec{d}_m), \end{aligned}$$

as wanted. □

The proof of the next proposition is completely similar to the classical one by Lawvere [61]. We sketch its main steps for the convenience of the reader.

For $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, a **B**-functor $U : \mathbf{K} \to \mathbf{B}$ will be said *tractable* if $\mathrm{nat}(U^n, U)$ $(= \mathbf{B}\ \mathrm{nat}(U^n, U)$ because 1 is a generator in **B**) is a set, where $U^n := \mathbf{K} \xrightarrow{U} \mathbf{B} \xrightarrow{\mathbf{B}[n,-]} \mathbf{B}$, $n = \{0, 1, \ldots, n-1\}$ with discrete order. Notice that if a **B**-functor $U : \mathbf{K} \to \mathbf{B}$ has a **B**-left adjoint F, it is tractable and $\mathrm{nat}(U^n, U)$ has a **B**-object structure given by the **B**-Yoneda Lemma (35), (33), because:

$$\begin{aligned} \mathbf{B}\,\mathrm{nat}(U^n, U^m) &= \mathbf{B}\,\mathrm{nat}(\mathbf{B}[n, U-], U^m) \\ &\simeq \mathbf{B}\,\mathrm{nat}(\mathbf{K}[Fn, -], U^m) \simeq U^m(Fn). \end{aligned}$$

$\mathbf{BCat}{\downarrow}\mathbf{B}$ will denote the category with objects tractable **B**-functors over **B**, $U : \mathbf{K} \to \mathbf{B}$, and morphisms **B**-functors over **B**, $G : (\mathbf{K}, U) \to (\mathbf{K}', U')$ such that $U' \cdot G = U$.

4.7. Proposition (Structure-Semantics Adjointness Theorem). *For* $\mathbf{B} = \mathbf{Pos}$,

$\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, *the functor* $\mathrm{Sem}: \mathbf{BTh}^{\mathrm{op}} \to \mathbf{BCat}\dot{\downarrow}\mathbf{B}: (T \xrightarrow{H} T') \mapsto ((\mathbf{B}_{T'}, U_{T'}) \xrightarrow{B_H} (\mathbf{B}_T, U_T))$ *has a left adjoint* Str.

Proof. Given $U: \mathbf{K} \to \mathbf{B}$, tractable, $\mathrm{nat}(U^n, U)$ has a natural $\mathbf{B}$-object structure given by the ordering: $\alpha \geq \beta$ iff $\alpha A \geq \beta A$ $\forall A \in \mathbf{K}$ (in the case $\mathbf{B}$-right adjoint, it is not difficult to see that this is the ordering given by the $\mathbf{B}$-Yoneda Lemma). We can then define a $\mathbf{B}$-theory $\mathrm{Str}\, U$ by taking $\mathrm{Str}\, U(n, 1) := \mathrm{nat}(U^n, U)$, with projections $(\pi_i: n \to 1) := \mathbf{B}[\bar{i}, -] \cdot U$, $\bar{i} := 1 \to n : 0 \mapsto i$, and composition the one of natural transformations (notice that $\mathrm{nat}(U^n, U^m) \simeq \mathrm{nat}(U^n, U)^m$). This provides the left adjoint. As in the classical case we have a counit $\varepsilon T: \mathrm{Str}\,\mathrm{Sem}\, T \simeq T$, which is an isomorphism, because as $T(n, m) = T(n, 1)^m$, it is $T(n, -) \in \mathbf{B}_T$, and by the $\mathbf{B}$-Yoneda Lemma, for any $A \in \mathbf{B}_T$ we have:

$$\mathbf{B}_T[T(n, -), A] := \mathbf{B}\,\mathrm{nat}(T(n, -), A) \simeq A^n = \mathbf{B}[n, A] \simeq \mathbf{B}_T[F_T(n), A],$$

which, again by $\mathbf{B}$-Yoneda, forces $T(n, -) \simeq F_T(n)$. Consequently we have

$$(\mathrm{Str}\,\mathrm{Sem}\, T)(n, 1) := \mathrm{nat}(U_T^n, U_T) \simeq U_T(F_T(n)) \simeq T(n, 1)$$

as desired. The remaining details are also completely similar to those in the proof of Lawvere [61] (see [91], for a published expository reference). □

4.8. Corollary (Completeness Theorem). *For* $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$:

(i) Sem *makes* $\mathbf{BTh}^{\mathrm{op}}$ *equivalent to a full reflective subcategory of* $\mathbf{BCat}\dot{\downarrow}\mathbf{B}$.

(ii) *Given* $\alpha, \beta: n \to 1$ *in a* $\mathbf{B}$*-theory* T, *it is* $\alpha \geq \beta$ *iff* $A\alpha \geq A\beta$ *for each* $A \in \mathbf{B}_T$.

Proof. (i) follows from the isomorphism εT (see [70] IV.3, Theorem 1). (ii) also follows from that, by definition of Str. □

We shall now prove a characterization theorem for varieties of chain-complete algebras, which is analogous to the classical theorem of Birkhoff [16] for varieties of universal algebras.

4.9. Birkhoff Variety Theorem. *For* $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, *and* $\mathscr{C}$ *a full subcategory of* Σ*-algebras on* $\mathbf{B}$, $\mathscr{C} \subseteq \mathbf{B}_\Sigma$, $\mathscr{C}$ *is a variety (more precisely:* $\mathscr{C}$ *is of the form* $\mathscr{C} = \mathbf{B}_H(\mathbf{B}_T)$ *for* $H: \bar{D}T_\Sigma \to T$ *a presentation of a* $\mathbf{B}$*-theory* T*) if and only if*

(i) *for* $\mathbf{B} = \mathbf{Pos}$, $\mathscr{C}$ *is closed in* $\mathbf{B}_\Sigma$ *under products, full subalgebras and surjective quotients;*

(ii) *for* $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$, $\mathscr{C}$ *is closed in* $\mathbf{B}_\Sigma$ *under products, persistently complete subalgebras, dense quotients and filtered colimits;*

(iii) *for* $\mathbf{B} = \mathbf{Pos}(\omega)$, $\mathscr{C}$ *is closed in* $\mathbf{B}_\Sigma$ *under products, full subalgebras, strongly dense quotients and filtered colimits.*

In addition, for $\mathbf{B} = \mathbf{Pos}(\omega)$, *if all algebras in* $\mathscr{C}$ *have a bottom element which is preserved by all* Σ*-homomorphisms in* $\mathscr{C}$, *then* $\mathscr{C}$ *is a variety if and only if it is closed in* $\mathbf{B}_\Sigma$ *under products, full subalgebras and strongly dense quotients.*

Proof. The case $\mathbf{B} = \mathbf{Pos}$ has been proved by Bloom in [19]. For $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$ resp. $\mathbf{Pos}(\omega)$, the "only if" part is 4.3. For the "if" part, let us consider as in the proof of 4.1 the "full-image factorization" $\bar{D}T_\Sigma \xrightarrow{A_0} T^0_A \xrightarrow{A^0} \mathbf{B}$ of each algebra $A \in \mathscr{C}$. Then denote $\bar{D}T_\Sigma \xrightarrow{\check{A}} \check{T}_A \hookrightarrow T_A$ the dense-persistently complete, resp. strongly dense-full mono, factorization of A in $\mathbf{BTh}$. Now, $\mathbf{BTh}$ is dense-cowellpowered, resp. strongly dense cowellpowered (see comments at the end of Section 3; notice also that $\mathbf{BTh}$, being locally presentable, is cowellpowered [42], 7.14). The dense, resp. strongly dense, quotients of $\bar{D}T_\Sigma$ thus form a complete lattice, and we can consider

$$(Q : \bar{D}T_\Sigma \to T_{\mathscr{C}}) := \sup\{(\check{A} : \bar{D}T_\Sigma \to \check{T}_A) \mid A \in \mathscr{C}\}$$

By the definition of Q, we certainly have $\mathscr{C} \subseteq \mathbf{B}_Q(\mathbf{B}_{T_\mathscr{C}})$. The rest of the proof will be to show that they actually coincide.

As for any $A \in \mathbf{B}_{T_\mathscr{C}}$ we have the counit $\varepsilon_{T_\mathscr{C}} A : F_{T_\mathscr{C}} U_{T_\mathscr{C}} A \to A$, which is a retraction in $\mathbf{B}$ (in particular dense, resp. strongly dense, homomorphism), it will be enough to show $F_{T_\mathscr{C}} B \in \mathscr{C}$ for any $B \in \mathbf{B}$. But we can reduce to show $F_{T_\mathscr{C}} DX \in \mathscr{C}$ for any $X \in \mathbf{Set}$, because in the diagram

$$\begin{tikzcd} F_\Sigma DVB \arrow[r, "F_\Sigma \varepsilon B"] \arrow[d, "r"'] & F_\Sigma B \arrow[d, "r'"] \\ F_{T_\mathscr{C}} DVB \arrow[r, "F_{T_\mathscr{C}} \varepsilon B"'] & F_{T_\mathscr{C}} B \end{tikzcd}$$

the reflection maps r, r' are dense, resp. strongly dense by 4.4(a), and $F_\Sigma \varepsilon B$ is bijective, as follows from the remarks on F_Σ in Section 3; hence $F_{T_\mathscr{C}} \varepsilon B$ is dense, resp. strongly dense.

We shall prove first a lemma:

4.10. Lemma. *If $\mathscr{C}$ is closed in $\mathbf{B}_\Sigma$ under products and persistently complete subalgebras, resp. products and full subalgebras, then $F_{T_\mathscr{C}}(DX) \in \mathscr{C}$, for any finite set X.*

Proof. Let $\mathbf{B} = \mathbf{Pos}(\dot{\omega})$. First notice that by 4.7, if $|X| = n$, it is $F_{T_\mathscr{C}} DX \simeq T_\mathscr{C}(n, 1)$, with operations $T_\mathscr{C}(n,1)(\alpha) := T_\mathscr{C}(n,\alpha) : T_\mathscr{C}(n,m) = T_\mathscr{C}(n,1)^m \to T_\mathscr{C}(n,1) : (\beta_1, \dots, \beta_m) \mapsto \alpha(\beta_1, \dots, \beta_m)$. Hence taking $n := \{0, 1, \dots, n-1\}$, the map $n \to T_\mathscr{C}(n,1) : i \mapsto \pi_i$ is universal, and for any "n-tuple" $(a_0, \dots, a_{n-1}) : n \to A$, to a $T_\mathscr{C}$-algebra A, the unique homomorphism is precisely $\overline{(a_0, \dots, a_{n-1})} : T_\mathscr{C}(n,1) \to A : \alpha \mapsto A\alpha(a_1, \dots, a_n)$. Now notice that

(i) If $\alpha, \beta : n \to 1$ in T and $\alpha \not\geq \beta$, there exists an algebra $A_{[\alpha,\beta]} \in \mathscr{C}$ such that $A_{[\alpha,\beta]}\alpha \not\geq A_{[\alpha,\beta]}\beta$. Because if no such $A_{[\alpha,\beta]}$ exists, i.e. $A\alpha \geq A\beta$ $\forall A \in \mathscr{C}$, and $\bar{R}_{[\alpha,\beta]}$ is the smallest theory congruent (many sorted) preorder on $T_\mathscr{C}$ containing the sequence of subsets $(R_{[\alpha,\beta]})_m := \{(\alpha, \beta)\}$ if $m = n$, $\varnothing$ otherwise, $n \in \omega$, then, by the many-sorted version of 3.2, all the dense theory quotients $\check{A} : \bar{D}T \to \check{T}_A$, $A \in \mathscr{C}$, factor through the dense quotient $DT \xrightarrow{Q} T_{\mathscr{C}} \to (T_{\mathscr{C}}/\bar{R}_{[\alpha,\beta]})$, against the definition of Q.

(ii) If $\{\alpha_m\} \subseteq T_{\mathscr{C}}(n,1)$ is a chain which has no upper bound, there exists an algebra $A_{[\{\alpha_m\}]}$ in $\mathscr{C}$ such that $\{A_{[\{\alpha_m\}]}\alpha_m\}$ has no upper bound in $\dot{\omega}[A^n_{[\{\alpha_m\}]}, A_{[\{\alpha_m\}]}]$ or, equivalently, in $T_{A_{[\{\alpha_m\}]}}(n,1)$. Because if no such $A_{[\{\alpha_m\}]}$ exists, i.e. there exists $\bigsqcup_m A\alpha_m$ in $T_A(n,1)$, for each $A \in \mathscr{C}$, then taking the family $\mathscr{F} = \{\mathscr{F}_k\}_{k\in\omega}$, with $\mathscr{F}_k := \{\alpha_m\}$ if $k = n$, $\varnothing$ otherwise; by the many-sorted version of 3.10, we have that all the dense theory quotients $\check{F} : \bar{D}T_\Sigma \to \check{T}_A$, $A \in \mathscr{C}$, factor through the dense theory quotient $\bar{D}T_\Sigma \xrightarrow{Q} T_{\mathscr{C}} \xrightarrow{\eta_{\mathscr{F}}} (T_{\mathscr{C}})_{\mathscr{F}}$, against the definition of Q.

Now, let $I_n = \{(\alpha,\beta) \mid \alpha,\beta \in T_{\mathscr{C}}(n,1), \alpha \not\geq \beta\} \cup \{\{\alpha_m\} \mid \{\alpha_m\}$ chain with no upper bound in $T_{\mathscr{C}}(n,1)\}$. For each $i \in I_n$, there exists a n-tuple $(a^i_0, \dots, a^i_{n-1}) : n \to A_{[i]}$ such that, (a) if $i = (\alpha,\beta)$, then $A_{[i]}\alpha(a^i_0,\dots,a^i_{n-1}) \not\geq A_{[i]}\beta(a^i_0,\dots,a^i_{n-1})$; (b) if $i = \{\alpha_m\}$, then $\{A_{[i]}\alpha_m(a^i_0,\dots,a^i_{n-1})\}$ has no upper bound in $A_{[i]}$. Consequently we have homomorphisms $\overline{(a^i,\dots,a^i_{n-1})} : T_{\mathscr{C}}(n,1) \to A_{[i]}$, $i \in I_n$, inducing a homomorphism

$$(\overline{(a^i_0,\dots,a^i_{n-1})})_{i\in I_n} : T_{\mathscr{C}}(n,1) \to \prod_i A_{[i]}$$

which is full and persistently complete by construction, which shows $T_{\mathscr{C}}(n,1) \in \mathscr{C}$, as desired.

The case $\mathbf{B} = \mathbf{Pos}(\omega)$ is completely analogous (but slightly simpler, because all chains are now bounded), and is left as an exercise. $\square$

Observe now that any set X is the filtered colimit of (the diagram of inclusions of) its finite subsets: $X = \operatorname{colim}\{X' \mid X' \subseteq X, |X'| \in \omega\}$. Consequently we have

$$F_{T_{\mathscr{C}}}DX = \operatorname{colim}\{F_{T_{\mathscr{C}}}DX' \mid X' \subseteq X, |X'| \in \omega\},$$

filtered colimit in $\mathbf{B}_{T_{\mathscr{C}}}$, hence also filtered colimit in $\mathbf{B}_\Sigma$, by 4.3 and 4.4(a), which by above lemma and previous remarks proves the "if" part.

We have left the case $\mathscr{C} \subseteq \mathbf{Pos}(\omega)_\Sigma$, all algebras in $\mathscr{C}$ with a bottom element, preserved inside $\mathscr{C}$ by all Σ-homomorphism, and $\mathscr{C}$ closed in $\mathbf{Pos}(\omega)_\Sigma$ under products, subalgebras and strongly dense quotients.

First notice that any set X with $|X| \geq \omega$, is the ω_1-filtered (42) colimit of (the diagram of inclusions of) its countable subsets:

$$X = \operatorname{colim}\{X' \mid X' \subseteq X, |X'| = \omega\}.$$

Hence $F_{T_{\mathscr{C}}}DX = \operatorname{colim}\{F_{T_{\mathscr{C}}}DX' \mid X' \subseteq X, |X'| = \omega\}$ is ω_1-filtered colimit in $\mathbf{Pos}(\omega)_{T_{\mathscr{C}}}$ and, by 4.6, also in $\mathbf{Pos}$. As the forgetful functor $V : \mathbf{Pos} \to \mathbf{Set}$ preserves filtered colimits (see for instance [80]), and having into account functoriality and the axiom of choice, we find that $F_{T_{\mathscr{C}}}DX$ is, set-theoretically, the union of the $F_{T_{\mathscr{C}}}DX'$, i.e.,

$$VF_{T_{\mathscr{C}}}DX = \bigcup\{VF_{T_{\mathscr{C}}}DX' \mid X' \subseteq X, |X'| = \omega\}.$$

Now, by 4.3, $\mathscr{C}$ is closed in $\mathbf{B}_{T_{\mathscr{C}}}$ under products and full subalgebras, and hence it is a reflective subcategory with strongly dense homomorphisms as reflections, [54], 37.1. As a consequence, it will be enough to prove that $F_{T_{\mathscr{C}}}DX' \in \mathscr{C}$, for $|X'| = \omega$, because given an arbitrary set X, for any countable $X' \subseteq X$ and denoting the inclusion by j we will have:

$$\begin{tikzcd} F_{T_\mathscr{C}}DX \arrow[r, "r"] \arrow[d, "F_{T_\mathscr{C}}Dj"'] & RF_{T_\mathscr{C}}DX \arrow[d, "RF_{T_\mathscr{C}}Dj"] \\ F_{T_\mathscr{C}}DX' \arrow[r, "\cong", "r'"'] & RF_{T_\mathscr{C}}DX' \end{tikzcd}$$

which by the previous remarks will show that r is a full monomorphism, given that $RF_{T_\mathscr{C}}Dj$ is a section by functoriality and the axiom of choice. Notice that r is also strongly dense, so it will be an isomorphism.

By Lemma 4.10 above we have also $T_\mathscr{C}(n, 1) \in \mathscr{C}$, $n \in \omega$. Hence there is a constant $\perp \in T_\mathscr{C}(0, 1)$ such that $\mathrm{id}_1 \geq \perp \bullet !$ in $T_\mathscr{C}(1, 1)$, for $! : 1 \to 0$ the terminal morphism in $T_\mathscr{C}$. So, as equalizers are full monomorphisms, $\mathscr{C}$ is closed in $\mathbf{B}_{T_\mathscr{C}}$ under limits and all has been now reduced to prove:

4.11. Lemma. *Let T be a* $\mathbf{Pos}(\omega)$*-theory with a constant $\perp$ such that* $\mathrm{id}_1 \geq \perp \bullet !$. *Then $F_TD(\omega)$ is the limit of the diagram:*

$$F_TD(0) \xleftarrow{\bar{h}_0} F_TD(1) \xleftarrow{\bar{h}_1} F_TD(2) \cdots F_TD(n) \xleftarrow{\bar{h}_n} F_TD(n+1) \cdots$$

where $\bar{h}_n$ is induced by $h_n : n+1 \to F_TD(n) : k \mapsto k$ if $0 \leq k < n$, $\perp$ if $k = n$.

Proof. Essentially this goes all the way back to Scott's Lattice of Flow Diagrams [94] and is the limit = colimit result of his D_∞ construction [95]. In other words, we have homomorphisms $F_TDj_n : F_TD(n) \to F_TD(n+1)$ for the inclusion maps $j_n : n \to n+1 : k \mapsto k$, $0 \leq k < n$, and consequently the equations

$$\bar{h}_n \bullet (F_TDj_n) = \mathrm{id}_{F_TD(n)}; \qquad (F_TDj_n)\bar{h}_n \leq \mathrm{id}_{F_TD(n+1)}$$

which guarantee (see [65, 97, 105, 106], for the $\mathbf{Pos}(\omega)_\perp$-version of Scott's result) that the limit of the diagrams with the $\bar{h}_n$'s is equal to the filtered colimit of the diagram with the F_TDj_n's, which by F_TD left adjoint is $F_TD(\omega)$. □□

4.12. Corollary. *For* $\mathbf{B} = \mathbf{Pos}$, *resp.* $\mathbf{Pos}(\dot\omega)$, *resp.* $\mathbf{Pos}(\omega)$, *and T, $T' \in$* $\mathbf{BTh}$, *a functor over* $\mathbf{B}$, *$G : \mathbf{B}_T \to \mathbf{B}_{T'}$, $G \bullet U_{T'} = U_T$, is a full embedding and makes $G(\mathbf{B}_T)$ full subcategory closed in $\mathbf{B}_{T'}$ under*

(i) *products, full subalgebras and surjective quotients, for* $\mathbf{B} = \mathbf{Pos}$;

(ii) *products, persistently complete subalgebras, dense quotients and filtered colimits for* $\mathbf{B} = \mathbf{Pos}(\dot\omega)$;

(iii) *products, full subalgebras, strongly dense quotients and filtered colimits for* $\mathbf{B} = \mathbf{Pos}(\omega)$ *if and only if G is of the form: $G = \mathbf{B}_H$, for $H : T' \to T$ a surjective, resp. dense, resp. strongly dense,* $\mathbf{B}$*-theory morphism.*

Proof. Reason on a presentation of T', and put together 4.3, 4.8(i), 4.9 and the remarks on $\mathbf{B}_H$ after Lemma 4.2. □

5. Semantics of computation

Some parts of this section are of an expository nature. Without any attempt to be complete, we try to show, in some instances, how the content of the previous sections relates to the semantics of programming languages and to other work in that area (see the very complete bibliography and comments on the topic currently known as Algebraic Semantics, recently collected by Andréka and Németi [10]). Familiarity with order-theoretic approaches to the semantics of computation is not assumed. We first show how the interpretations of a programming language can be viewed as ω-complete algebras, in a way which automatically gives the semantics of the language in a functorial manner. Then we connect the varieties studied in the last section with the work of Courcelle, Guessarian and Nivat on algebraic classes of interpretations. Roughly speaking, for the problem of $\mathscr{C}$-equivalence between program schemas, the crucial information lies not in the smaller variety corresponding to the class $\mathscr{C}$, but in a nice variety (corresponding to a nice quotient theory) which is smaller in a certain sense. We then explain how algebras on the category $\mathbf{Pos}(\omega)$ appear in a natural way, when the interest is focused in the "computable" theory morphisms, and bring together, on the ground $\mathbf{Pos}(\omega)$, the concepts of rational theory and regular algebra, both naturally associated with rational program schemas. This solution seems to be different from the one previously given by Tyurin. We think that it will prove useful in connection with higher-order recursion.

In a programming language one normally has basic function symbols and predicate symbols, which are the basis for recursive definitions in the language. An *interpretation* of that language will provide a collection of sets corresponding to the basic types, and corresponding functions and predicates, in such a way that the recursive definitions at the level of the language will correspond to (partial) functions recursively defined from the basic functions and predicates. For simplicity we will assume, in what follows, that there is only one basic type in the language, hence only one set A in the interpretation. We can make all functions total, instead of partial, by allowing an extra "undefined" element in the set A that we shall denote $\perp \in A$. A function symbol f will have an arity $\#f \in \omega$ and in an interpretation will correspond to a map $Af: A^{\#f} \to A$. For instance if $A = \mathbb{N} \cup \{\perp\}$ we may have constants $0, 1, \ldots$, addition, subtraction and multiplication, extended in the obvious way in the undefined cases. Moreover, the "more defined" relation gives an ordering to A of the form $a \geq a'$ iff $a = a'$ or $a' = \perp$ (i.e. what is called a *flat* or "discrete" order on A), which will make the functions Af monotonic, and trivially continuous. Following McCarthy [74] and Nivat [84], we can see predicates also as functions. Namely if $Ap: A^n \to \{0, 1, \perp\}$ is a three valued predicate, we associate with it the function $A(p \to -, -): A^{n+2} \to A : (a_1, \ldots, a_n, a_{n+1}, a_{n+2}) \mapsto$ *if* $p(a_1, \ldots, a_n) = 1$ *then* a_{n+1} *else if* $p(a_1, \ldots, a_n) = 0$ *then* a_{n+2} *else if* $p(a_1, \ldots, a_n) = \perp$ *then* $\perp$. Hence calling Σ to the ranked alphabet formed by function symbols, predicate symbols (in the form $(p \to -, -)$) and the undefined symbol, $\perp$, an interpretation can be viewed as a ω-complete Σ-algebra $A : \bar{D}T_\Sigma \to \mathbf{Pos}(\omega)$, such that its order is the flat order on A mentioned before. The algebra A so defined satisfies the inequality $\mathrm{id}_1 \geq \perp \cdot !$. Let R denote the smallest congruent theory preorder generated by that inequality, and $\bar{D}T_\Sigma \xrightarrow{J} CT_\Sigma := (\bar{D}T_{\Sigma/R})$ the corresponding nice quotient map (note that the notation CT_Σ, adapted from what in [5] is used to denote the corresponding initial algebra, clashes somewhat with the one used before for the completion functor $\bar{C}$. However, the functor $\bar{C}$ is the identity on theories of the form $\bar{D}T$). Then we have a factorization

$$\begin{tikzcd} \bar{D}T_\Sigma \arrow[r, "J"] \arrow[dr, "A"'] & CT_\Sigma \arrow[d, dashed, "A"] \\ & \mathbf{Pos}(\omega) \end{tikzcd}$$

The free CT_Σ-algebras on discrete posets DX were investigated in [48] and [7], and denoted $CT_\Sigma(X)$ there. The elements of $CT_\Sigma(X)$ can be described as infinite Σ-trees (see [7] and 5.2 below) with constants and elements of X in the leaves, in the same way as elements of a word algebra can be described as finite trees. Besides, any infinite tree is the l.u.b. in the ordering of the chain of finite trees obtained considering the subtrees of depth n, for each n, formed by substituting $\perp$ on those leaves of depth n corresponding to nodes previously labeled by non-constant elements of Σ. This is very useful, because the "unfoldment" of a recursive definition (which can be thought of as its syntaxtical or symbolic execution) can be represented by an infinite tree [94, 7, 29]. $\mathbf{Pos}(\omega)_{CT_\Sigma}$ is as a consequence a category where syntax, in the form $CT_\Sigma(X)$, and semantics, in the form of the flat or "discrete" order interpretations discussed above, can be brought together as algebras, in such a way that the syntactic evaluation of a recursive definition is mapped to its actual evaluation or semantics in a functorial (and homomorphic) way. We shall illustrate this point with the simple example of the recursive definition of the factorial function $n!$:

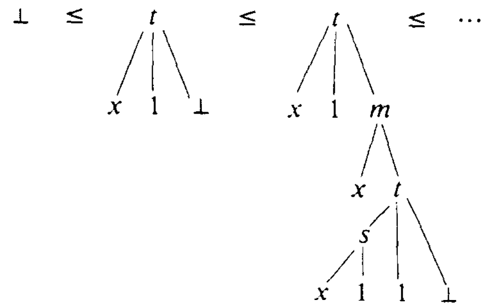
5.1. Example. Let $m, s \in \Sigma_2$, $t \in \Sigma_3$ be the symbols which are interpreted as multiplication, subtraction and "if $x_1 = 0$ then x_2 else x_3" in the interpretation $A = \mathbb{N} \cup \{\perp\}$ discussed before.

$$\varphi(x) = t(x, 1, m(x, \varphi(s(x, 1))))$$

induces a continuous function

$$\gamma : \omega[A, A] \to \omega[A, A]$$
$$f \mapsto \lambda x \cdot At(x, 1, Am(x, f(As(x, 1))))$$

whose minimal fixpoint, $\bigsqcup_n \gamma^n(\perp)$ (where $\perp$ denotes here the constant function $A \to A : x \mapsto \perp$) is the factorial function (i.e. the "semantics" of the recursive definition). We could as well have considered the "unfoldment" of the recursive definition, by means of repeated applications of it to the symbol $\perp$ ("copy rule") to get the chain of trees



This can be seen as a chain in $CT_\Sigma(1,1) \simeq CT(\{x\})$, and as in the algebra $A : CT_\Sigma \to \mathbf{Pos}(\omega)$ the finite trees of that chain are mapped to the functions $\gamma^n(\perp)$, by continuity, the infinite tree which is their limit has to be mapped to the factorial function. Of course, the mapping from trees to functions is a homomorphism, because $\omega[A,A]$ has the CT_Σ-algebra structure:

$$CT_\Sigma \xrightarrow{A} \mathbf{Pos}(\omega) \xrightarrow{\omega[A,-]} \mathbf{Pos}(\omega)$$

and the map $A_1 : CT_\Sigma(1,1) \to \omega[A,A]$ is just the unique homomorphism induced by the map $\{x\} \to \omega[A,A] : x \mapsto \mathrm{id}_A$.

More generally, for Σ as above, a *set of recursive equations* or *recursive program schema* can be defined as in [84], as a collection $\Lambda = (\varphi_i(x_1, \ldots, x_{\#\varphi_i}) = w_i)_{i=1,\ldots,k}$, where the φ_i are in an auxiliary ranked alphabet Ψ of "function variables", and each w_i is an element of the $\Sigma \cup \Psi$-word algebra on the letters $x_1, \ldots, x_{\#\varphi_i}$. Again as in 5.1 this induces for any CT_Σ-algebra A a continuous endofunction ν on $\prod_i \omega[A^{\#\varphi_i}, A]$, whose minimal fixpoint is the *semantics* of the program schema Λ in the "interpretation" A. Alternatively, the schema Λ can be seen as a tree rewriting system; then the set of finite trees of $CT_\Sigma(\{x_1, \ldots, x_{\#\varphi_i}\})$ generated starting with φ_i is *filtered* [84], and as a consequence has (by Iwamura's Lemma, cf. [75]) a least upper bound in $CT_\Sigma(\{x_1, \ldots, x_{\#\varphi_i}\}) \simeq CT_\Sigma(\#\varphi_i, 1)$. Again this l.u.b. is mapped by $A : CT_\Sigma \to \mathbf{Pos}(\omega)$ to the ith component of the minimal fixpoint $\bigsqcup_n \gamma^n(\perp)$ (cf. [84], where also the "operational semantics" (when the order of A is flat) is discussed, and [30]).

From now on, we shall adopt the viewpoint that *an "interpretation" of a programming language with basic operations Σ, is nothing else but a CT_Σ-algebra.* Then we can ask the question of *equivalence* between program schemas: when is the semantics of two program schemas Λ and Λ' the same for all interpretations A $\mathbf{Pos}(\omega)_{CT_\Sigma}$?. By the completeness theorem 4.8, this will be the case if and only if the infinite trees corresponding to "unfold" Λ and Λ' coincide in CT_Σ. To prove interesting properties about *programs* (i.e. program schemas Λ together with an interpretation A) and in particular equivalence, one has generally to assume more than just the fact that A is a CT_Σ-algebra. In a good number of cases the assumptions on A amount to the fact that A belongs to a *class* $\mathscr{C} \subseteq \mathbf{Pos}(\omega)_{CT_\Sigma}$ of interpretations. The class $\mathscr{C}$ needs not be a variety, but will always be contained in a variety which is smallest in a way to be made precise below. If such a variety has a theory T, and we can show that two program schemas are equivalent in T, by the completeness theorem, the corresponding programs will be equal in any interpretation belonging to $\mathbf{Pos}(\omega)_T$, and a fortiori in $\mathscr{C}$.

Note that by 4.9 and 4.12, we have, for $T \in \mathbf{Pos}(\omega)\mathbf{Th}$, a Galois connection between the complete lattice $\mathbf{FSC}(\mathbf{Pos}(\omega)_T)$ of full subcategories of $\mathbf{Pos}(\omega)_T$ and the complete lattice $\mathbf{SDQuot}(T)$ of strongly dense quotients of T, given by the maps: $\mathscr{C} \mapsto (T \to T_{\mathscr{C}})$; $(T \xrightarrow{Q} T') \mapsto \mathbf{Pos}(\omega)_Q(\mathbf{Pos}(\omega)_T)$. This gives an isomorphism between the complete lattice $\mathbf{Var}(\mathbf{Pos}(\omega)_T)$ of varieties of $\mathbf{Pos}(\omega)_T$ and the lattice of strongly dense quotients of T, in such a way that greatest lower bounds of quotients corresponds to *intersection* of subcategories. To see this, the only point which requires a little argument is the fact that the map $\mathscr{C} \mapsto (T \to T_{\mathscr{C}})$ is surjective. But this follows from the definition of $T_{\mathscr{C}}$ in the proof of 4.9 and the fact that, for any $T' \in \mathbf{Pos}(\omega)\mathbf{Th}$, the factorization (defined as in the proof of 4.9) $\check{F_{T'}DX} : T' \to \check{T}_{F_{T'}DX}$, of the free algebra $F_{T'}DX$, for X an infinite set, is an isomorphism, because, by the axiom of choice, the homomorphism

$$F_{T'}D(n) \xrightarrow{F_{T'}Dk} F_{T'}DX$$

(corresponding to an injective map $k : n \to X$) is a full monomorphism, and so is too the homomorphism

$$h := T'(n, -) \simeq F_{T'}D(n) \xrightarrow{F_{T'}Dk} F_{T'}DX.$$

As given $\alpha, \beta : n \to 1$ in T' it is $\alpha \geq \beta$ in T' iff $T'(n, \alpha) \geq T'(n, \beta)$, by h full monomorphism, we have: $F_T DX(\alpha) \geq F_T DX(\beta)$. This proves $\check{F_{T'}DX}$ above full mono, hence iso. (Note that, with slight modifications, the argument above also works for the cases $\mathbf{Pos}$ and $\mathbf{Pos}(\dot{\omega})$).

Returning to the isomorphism $\mathbf{Var}(\mathbf{Pos}(\omega)_T) \simeq \mathbf{SDQuot}(T)$, we can define a closure operator on $\mathbf{SDQuot}(T)$ by the rule: $(T \xrightarrow{Q} T') \mapsto (T \to (\widetilde{T/R_Q}))$, which sends each quotient Q to the nice quotient corresponding to the congruent preorder R_Q induced by Q. We shall call a variety of $\mathbf{Pos}(\omega)_T$ *nice* if it corresponds to a nice quotient of T or, equivalently, if it is definable as the class of algebras $A \in \mathbf{Pos}(\omega)_T$ satisfying $A\alpha \geq A\beta$ for $(\alpha, \beta) \in R_n \subseteq T(n, 1)^2$, for a given family $R = (Rn)$ of arbitrary relations. So we have an isomorphism $\mathbf{NVar}(\mathbf{Pos}(\omega)_T) \simeq \mathbf{NQuot}(T)$ between the complete lattices of nice varieties and of nice quotients such that the g.l.b.'s in $\mathbf{NQuot}(T)$ are those in $\mathbf{SDQuot}(T)$ and correspond to intersection of full subcategories in $\mathbf{NVar}(\mathbf{Pos}(\omega)_T)$. Of course we have also an antiisomorphism of $\mathbf{NQuot}(T)$ with the lattice of congruent theory preorders on T.

Coming back to classes interpretations $\mathscr{C} \subseteq \mathbf{Pos}(\omega)_{CT_\Sigma}$, the above considerations are particularly important in the cases $T = \bar{D}T_\Sigma$ and $T = CT_\Sigma$. The nice quotients of

$\bar{D}T_\Sigma$ have a very simple description, because all are of the form

$$\bar{D}T_\Sigma = \bar{C}\bar{D}T_\Sigma \xrightarrow{\bar{C}Q_R} \bar{C}(\bar{D}T_{\Sigma/R})$$

for $C = \wedge : \mathbf{Pos} \to \mathbf{Pos}(\omega)$, the "algebraic" completion or completion by ideals, and R a congruent theory $\mathbf{Pos}$-preorder on $\bar{D}T_\Sigma$. This observation is due to Courcelle–Guessarian–Nivat [50, 29, 30]. We shall call to these quotients *algebraic quotients*, and *algebraic varieties* to the corresponding varieties.

For the convenience of the reader we include a lemma whose ω-sorted version shows the property just mentioned:

5.2. Lemma. *Let $T \in \mathbf{Th}$ be an ordinary Lawvere theory, then for any algebra of the form $\bar{D}A \in \mathbf{Pos}(\omega)_T$, A an ordinary T-algebra, every nice quotient of $\bar{D}A$ in $\mathbf{Pos}(\omega)_T$ is of the form $\widehat{\bar{D}A} = \bar{D}A \xrightarrow{\hat{q}_R} \widehat{\bar{D}A/R}$ for R a congruent $\mathbf{Pos}$-preorder on A.*

Proof. *As $\bar{D}A$ is discrete, congruent $\mathbf{Pos}$-preorders and congruent $\mathbf{Pos}(\omega)$-preorders on A coincide. For any such preorder R the above map $\hat{q}_R$ has the desired universal property as follows from the diagram:*

$$\begin{tikzcd} \bar{D}A \arrow[r, "q_R"] \arrow[drr, "f"'] & \bar{D}A/R \arrow[r, "\eta"] \arrow[dr, dashed, "\bar{f}"] & \widehat{\bar{D}A/R} \arrow[d, "\hat{f}"] \\ & & B \end{tikzcd}$$

where $B \in \mathbf{Pos}(\omega)_T$, and f is a homomorphism such that $R_f \supseteq R$ (apply now 3.2 *and* 3.8(b), *for $\mathbf{B} = \mathbf{Pos}$, and* 4.2 *for the inclusion $\mathbf{Pos}(\omega) \hookrightarrow \mathbf{Pos}$ and the theory $\bar{D}T$, and note that q_R is surjective).* $\square$

The nice quotients of CT_Σ correspond to theory-congruent $\mathbf{Pos}(\omega)$-preorders on CT_Σ which in turn can be generated by an arbitrary family of relations $R = (R_n)_n$ $R_n \subseteq CT_\Sigma(n, 1)^2$. The corresponding nice varieties might be called *schematic varieties,* because include as particular instance the case where R_n consists of pairs of trees corresponding to "unfoldments" of schemas, but one should have into account that if Σ is, say, countable, there is only a countable number of trees in CT_Σ corresponding to unfoldments of schemas, whereas CT_Σ will contain an uncountable number of trees.

Let now $\mathscr{C} \subseteq \mathbf{Pos}(\omega)_{CT_\Sigma}$ be a class of interpretations (i.e. a full subcategory), then by the construction in the proof of 4.9 we have the map $Q : \bar{D}T_\Sigma \to T_{\mathscr{C}}$ which provides the smallest variety containing $\mathscr{C}$. As $\mathscr{C}$ is inside $\mathbf{Pos}(\omega)_{CT_\Sigma}$ the map Q factors as:

$$Q = DT_\Sigma \xrightarrow{J} CT_\Sigma \xrightarrow{Q'} T_{\mathscr{C}},$$

and as CT_Σ is a nice quotient of $\bar{D}T_\Sigma$, namely by imposing the relation $R_1 =$

$\{(\mathrm{id}_1, \perp \cdot !)\}$, we will have the diagram

$$\begin{tikzcd}
DT_\Sigma \arrow[r, "J"] \arrow[rr, bend left, "Q"] \arrow[dr, "G"'] & CT_\Sigma \arrow[r, "Q'"] \arrow[d, "H"] \arrow[dr, "H'"] & T_{\mathscr{C}} \\
 & T_1 := (\bar{D}T_\Sigma/R_Q) \arrow[r] & (CT_\Sigma/R_{Q'}) := T'_1 \arrow[u]
\end{tikzcd}$$

and consequently a chain of full embeddings

$$\mathscr{C} \to \mathbf{Pos}\,(\omega)_{T_{\mathscr{C}}} \to \mathbf{Pos}\,(\omega)_{T'_1} \to \mathbf{Pos}\,(\omega)_{T_1} \to \mathbf{Pos}\,(\omega)_{CT_\Sigma}$$

where, by construction, $\mathbf{Pos}\,(\omega)_{T_1}$ is the smallest algebraic variety containing $\mathscr{C}$ and $\mathbf{Pos}\,(\omega)_{T'_1}$ the smallest schematic variety containing $\mathscr{C}$. Concerning the problem of equivalence of schemas, if we have into account the remark (i) in the proof of 4.9 and consider the surjective-full mono factorization of Q' in **PosTh** we have the following: given two program schemas A, A' with unfoldments $\alpha_1, \ldots, \alpha_n$; $\alpha'_1, \ldots, \alpha'_n$, in CT_Σ, such that α_i and α'_i are both of the same arity, then $H'\alpha_i \geq H'\alpha'_i$ in T'_1, $i = 1, \ldots, n$, if and only if $A\alpha_i \geq A\alpha'_i$ for every $A \in \mathscr{C}$. If, in addition, R'_Q has the property of being the smallest congruent theory preorder generated by $J^2(R_Q)$, then we will have $R_H = R'_Q$, and consequently the map $T_1 \to T'_1$ above will be an isomorphism, because it is immediate to check that H is a nice quotient of CT_Σ (by J epi and G nice). Hence in this latter case we will have $H\alpha_i \geq H\alpha'_i$ iff $A\alpha_i \geq A\alpha'_i$, $i = 1, \ldots, n$. In this latter case the class $\mathscr{C}$ is called an *algebraic class of functional interpretations* in [30]. The great importance of having a class $\mathscr{C}$ which is algebraic is that as T_1 is the algebraic completion of $\bar{D}T_{\Sigma/R_Q}$, given two infinite trees in $CT_\Sigma(n, 1)$, $\alpha = \bigsqcup \alpha_n$, $\beta = \bigsqcup \beta_m$, where $\{\alpha_n\}$, $\{\beta_m\}$ are chains of finite trees approximating α and β, it will be:

$$\begin{aligned} H\alpha \geq H\beta \quad &\text{iff} \quad \forall m \in \omega\ \exists n \in \omega \text{ such that } H\alpha_n \geq H\beta_m, \\ &\text{iff} \quad \forall m \in \omega\ \exists n \in \omega \text{ such that } \alpha_n\, R_Q\, \beta_m. \end{aligned}$$

This is a very good situation, in which proofs of equivalence between program schemas can be obtained by means of induction principles, or tree rewriting systems [26]. See [30, 26] and the comments and references there.

We have already pointed out that CT_Σ contains many elements which are "non-computable". There are good reasons to try to isolate "computable" subtheories of CT_Σ. Among other things, this can help to understand the nature of higher order recursion [32]. Recursive schemas are the second step in a hierarchy in which *rational* schemas are at the same time the first step and, in a sense, the way to pass from step n to step $n+1$ [107, 71, 41, 32, 8]. Our opinion—one might even say hunch—is that a better understanding of higher order recursion will be gained from an adequate answer to the question of what the *algebras* of a *rational theory* [3] are.

(Close relatives of rational theories are *μ-clones* and *iterative theories,* cf. [104, 102, 103, 13, 40]). The answer that we propose uses **Pos** $(\dot{\omega})$ as base category, and seems to be different from the solution given by Tyurin [100, 101]. Consequences of what follows will appear elsewhere. We begin with some motivation:

A *rational* (Σ) *schema,* or *system of regular equations with parameters* is a list

$$\Lambda : \begin{cases} x_1 = w_1(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+p}) \\ \vdots \\ x_n = w_n(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+p}) \end{cases}$$

where $\alpha_1 = w_1(x_1 \cdots x_{n+p}), \dots, \alpha_n = w_n(x_1, \dots, x_{n+p})$, are words in the Σ-word algebra on the letters $x_1, \dots, x_{n+p}$. For any CT_Σ-algebra A the semantics, $A(\alpha^\dagger)$, of Λ is defined in the following way: let $\alpha : n+p \to n$ denote the morphism $(\alpha_1, \dots, \alpha_n)$ in CT_Σ; then $A(\alpha^\dagger)$ is the function in the diagram

$$\begin{tikzcd} A^p \arrow[r, "\overline{A(\alpha)}"] \arrow[rr, bend right, "A(\alpha^\dagger)"'] & \omega[A^n, A^n] \arrow[r, "\mu"] & A^n \end{tikzcd}$$

where $\overline{A(\alpha)} := \lambda y \bullet A(\alpha)(x, y)$, and μ sends each continuous function to its minimal fixpoint.

Each rational schema can be made into a recursive schema, so that there is a morphism $\alpha^\dagger : p \to n$ in CT_Σ which is the tuple of its unfoldments and which is mapped by A to $A(\alpha^\dagger)$, as above. However a simpler description of $\alpha^\dagger$ is possible [4]:

$$\alpha^\dagger = \pi_n^{n+p} \bullet \alpha^\nabla$$

where $\pi_n^{n+p} := (\pi_1, \dots, \pi_n) : m+p \to n$, $\pi_p^{n+p} := (\pi_{n+1}, \dots, \pi_{n+p}) : n+p \to p$, denote the projections to n and p in CT_Σ and $\alpha^\nabla := \bigsqcup_m \alpha^{(m)}$, where

$$\alpha^{(m)} := p \xrightarrow{(\perp, \dots, \perp, \mathrm{id}_p)} n+p \xrightarrow{(\alpha, \pi_p^{n+p})^m} n+p$$

(Note that the construction α^∇ can be recovered from $\alpha^\dagger$ as: $\alpha^\nabla = (\alpha^\dagger, \mathrm{id}_p)$).

The compositions of finite trees of CT_Σ with trees of the form $\alpha^\dagger$ associated to rational schemas forms a sub-theory RT_Σ of CT_Σ which is closed under the operation $\dagger$, i.e. we can replace the finite trees $\alpha_1, \dots, \alpha_n$ by trees in RT_Σ and get another tree in RT_Σ [3, 4]. According to our previous viewpoint, the homsets $RT_\Sigma(n, 1)$ should be in some way the finitely generated free RT_Σ-algebras and provide the "abstract syntax" for rational schemas. But $RT_\Sigma(n, 1)$ is not ω-complete so, for instance, we cannot use the minimal fixpoint operator μ as before for defining $A(\alpha^\dagger)$. We need some definitions.

5.3. Definition. For $\mathbf{B} = \mathbf{Pos}$, $\mathbf{Pos}(\dot{\omega})$, $\mathbf{Pos}(\omega)$, a $\mathbf{B}$-theory T is said a *strict* $\mathbf{B}$*-theory* if there is a constant $\perp \in T(0, 1)$ such that $\mathrm{id}_1 \geq \perp \bullet !$. Of course the constant $\perp$ is then unique. A morphism of strict $\mathbf{B}$-theories is a theory morphism which preserves $\perp$, so we have a category that we shall denote $\mathbf{BTh}_\perp$. (From the definition it is easy to see that $\mathbf{BTh}_\perp$ is a category of ω-sorted algebras for a **Pos**-theory of "strict theories" $T_{\mathrm{Th}_\perp}$, and that the inclusion functor $\mathbf{BTh}_\perp \hookrightarrow \mathbf{BTh}$ is of the form $\mathbf{B}_J$ for $J : \bar{D}T_{\mathrm{Th}} \to T_{\mathrm{Th}_\perp}$ the obvious map from the theory of theories to the theory of strict theories).

5.3. Definition (ADJ [3]). Let $T \in \mathbf{PosTh}_\perp$. T is said a *rational* theory iff for each $\alpha : n+p \to n$ in T

(i) The chain $\{\alpha^{(m)}\}$, defined as above has a l.u.b.

$$\alpha^{\nabla} := \bigsqcup_m \alpha^{(m)} \quad \text{in } T(p, n'+p).$$

(ii) Whenever the composition is defined it is

$$\beta \bullet \alpha^{\nabla} = \bigsqcup_m \beta \bullet \alpha^{(m)}, \qquad \alpha^{\nabla} \bullet \gamma = \bigsqcup_m \alpha^{(m)} \bullet \gamma.$$

$H : T \to T'$ is a map of rational theories if it is a map of strict theories and satisfies $H(\alpha^{\nabla}) = (H\alpha)^{\nabla}$ for any α as above.

We shall say that a theory $T \in \mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$ is a *rational* $\mathbf{Pos}(\dot{\omega})$*-theory* if it is a rational theory (note that condition (ii) above is satisfied by definition). Rational $\mathbf{Pos}(\dot{\omega})$-theories form a full subcategory of $\mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$ that we shall denote $\mathbf{Rat\,Pos}(\dot{\omega})\mathbf{Th}$.

5.4. Remark. It can be shown that $\mathbf{Rat\,Pos}(\dot{\omega})\mathbf{Th}$ is a variety of $\mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$, corresponding to a quotient of $T_{\mathrm{Th}_\perp}$ of the form $\eta_{\mathscr{F}} : T_{\mathrm{Th}} \to T_{\mathrm{Th}_{\mathscr{F}}}$ as in 3.10. We shall describe its reflection maps soon.

5.5. Definition. Let $T \in \mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$. Then an algebra $A \in \mathbf{Pos}(\dot{\omega})_T$ is said a *regular T-algebra* if for any $\alpha : n+p \to n$ in T there exists a l.u.b. $\bigsqcup_n A(\alpha^{(n)})$ in $\dot{\omega}[A^p, A^{n+p}]$. Regular algebras form a full subcategory of $\mathbf{Pos}(\dot{\omega})_T$, that we shall denote $\mathbf{Reg\,Pos}(\dot{\omega})_T$.

We shall now make use of a doubly generalized version of 3.10 to "impose freely" the l.u.b.'s α^{∇} in the theory T. As mentioned in 4.4(b) we can replace the ordinary Lawvere theory T which appears in 3.10 by a $\mathbf{Pos}(\dot{\omega})$-theory and 3.10 is still valid. Generalize now to ω-sorted algebras and theories and apply this version of 3.10 replacing T, A and $\mathscr{F}$ there by the following: $T := \bar{C}T_{\mathrm{Th}_\perp}$ ($C = \dot{\wedge} : \mathbf{Pos} \to \mathbf{Pos}(\dot{\omega})$), $A := T \in \mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$ and $\mathscr{F} := (\mathscr{F}_p)$, $p \in \omega$, with $\mathscr{F}_p := \{\{\pi_i \bullet \alpha^{(m)}\}_m \mid \alpha : n+p \to n \text{ in } T, i = 1, \ldots, n+p\}$. We shall use in what follows the notation $\eta_T : T \to RT$ for the map $\eta_{\mathscr{F}} : T \to T_{\mathscr{F}}$, so defined.

5.6. Proposition. *For* $T \in \mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$, *the category* $\mathbf{Reg\,Pos}(\dot{\omega})_T$ *of its regular algebras is a variety, namely the one corresponding to the dense theory map* $\eta T : T \to RT$ *defined above. Besides, the reflection maps* $A \to \mathrm{Reg}\,A$, $A \in \mathbf{Pos}(\dot{\omega})_T$, *for the inclusion of regular algebras into T-algebras, are full monomorphisms.*

Proof. For any regular algebra A, take the (strict) theory morphism $T \xrightarrow{A_0} T_A^0$ in its full-image factorization (see the proof of 4.1). By definition of regular algebra, all the chains which are image by A of the ones in $\mathscr{F}$, above, will have a limit in T_A^0. So we get a unique induced $\bar{A}: RT \to T_A^0$, which makes A into a RT-algebra. As a consequence $\mathbf{Reg\,Pos}(\dot{\omega})_T \hookrightarrow \mathbf{Pos}(\dot{\omega})_{RT}$. On the other hand, any RT-algebra is (considered as a T algebra by composing with η_T) a regular T-algebra, by definition of RT and $\mathbf{Pos}(\dot{\omega})$-functoriality of the algebra considered as a functor. This proves the first assertion. To see that the (dense) reflection maps $A \to \operatorname{Reg} A$ into the variety $\mathbf{Reg\,Pos}(\dot{\omega})_T$ are full monomorphisms, note that by construction (cf. 3.10) we have the factorization

$$\begin{tikzcd} T \arrow[rr, "\bar{\eta}_T"] \arrow[dr, "\eta_T"'] & & \bar{T} \\ & RT \arrow[ur] & \end{tikzcd}$$

where the three maps are dense. So we have full embeddings

$$\mathbf{Pos}(\dot{\omega})_T \leftarrow \mathbf{Reg\,Pos}(\dot{\omega})_T \leftarrow \mathbf{Pos}(\dot{\omega})_{\bar{T}} \leftarrow \mathbf{Pos}(\omega)_{\bar{T}} = \mathbf{Pos}(\omega)_T$$

where the two embeddings on the left have reflections because they are varieties, the composition of all them has full monomorphic reflection maps, $A \xrightarrow{\bar{\eta}_A} \bar{A}$, by 4.2, and the full embedding $\mathbf{Pos}(\omega)_{\bar{T}} \to \mathbf{Pos}(\dot{\omega})_{\bar{T}}$ is reflective as can be seen by a triangle argument, or by the fact that $\mathbf{Pos}(\omega)_{\bar{T}}$ is closed in $\mathbf{Pos}(\dot{\omega})_{\bar{T}}$ under products and persistently complete subalgebras, which makes it dense-reflective subcategory [54]. So we have a factorization of reflection maps

$$\begin{tikzcd} A \arrow[rr, "\bar{\eta}_A"] \arrow[dr] & & A \\ & \operatorname{Reg} A \to \mathbf{Pos}(\dot{\omega})_{\bar{\eta}_T(A)} \arrow[ur] & \end{tikzcd}$$

which forces $A \to \operatorname{Reg} A$ to be full monomorphism. □

5.7. Proposition. *For any $T \in \mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$, the theory RT is a rational* $\mathbf{Pos}(\dot{\omega})$-*theory. The maps $\eta_T: T \to RT$, $T \in \mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$, provide a reflection for the inclusion* $\mathbf{Rat\,Pos}(\dot{\omega})\mathbf{Th} \hookrightarrow \mathbf{Pos}(\dot{\omega})\mathbf{Th}_\perp$.

Proof. The second assertion follows, if we prove the first, by the universal property that the map η_T has, by definition. As for the first assertion, note that we have the factorization $T \xrightarrow{\eta_T} RT \hookrightarrow \bar{T}$, where both maps are full monomorphisms (and we shall consider the first an inclusion to simplify notation). As RT contains the morphisms α^∇ for all $\alpha: n+p \to n$ in T, it will also contain all morphisms of the form $\beta \cdot \alpha^\nabla$, β, α in T. Call R^0T to the collection of those $\beta \cdot \alpha^\nabla$. It is a theorem of ADJ [3], Th. 5.8 that R^0T is a rational theory, with the theory operations induced by those of T. As RT coincides by definition with the smallest full (strict) **Pos**$(\dot\omega)$ subtheory of $\bar{T}$ containing R^0T we can reduce to prove

5.8. Lemma. *Let $T \hookrightarrow T'$ be a full rational subtheory of a* **Pos**(ω)*-theory T', and let $\bar{T} = \{\bar{T}(n,1) \mid n \in \omega\}$ be the family of $\dot\omega$-complete posets which are the strongly dense closure of the subsets $T(n,1) \hookrightarrow T'(n,1)$ in the category* **Pos**$(\dot\omega)$. *Then T is a full rational* **Pos**$(\dot\omega)$*-subtheory of T'.*

Proof. We recall that for $B \in$ **Pos**$(\dot\omega)$ the strongly dense closure $\bar{A}$ of a subset $A \subseteq B$ is the intersection of all full subobjects of B containing A. $\bar{A}$ can be generated transfinitely as $\bar{A} = \bigsqcup_\gamma A_\gamma$ with $A_0 = A$, $A_{\gamma+1} = A_\gamma \cup \{\bigsqcup a_n \mid \{a_n\}$ chain bounded in $A_\gamma\}$, and for limit ordinals $A_\gamma = \bigcup_{\gamma'<\gamma} A_{\gamma'}$. The lemma will be proved if we show that $T_\gamma = \{T_\gamma(n,1) \mid n \in \omega\}$ is a theory and is rational for each γ (notice that condition (ii) 5.3 will be automatic by continuity of composition in T'). But the assertion follows by transfinite induction from the following:

(i) For $\gamma = 0$, $T_0 = T$ is rational.

(ii) If T_γ is rational, $T_{\gamma+1}$ is rational because:

(a) If $\{\alpha_k\}$ is a chain in $T_\gamma(n+p,n)$ bounded by, say $\beta : n+p \to n$ in T, then it is easy to check that the chain $\{\alpha_k^\nabla\}$ is bounded by β^∇ in $T_\gamma(p, n+p)$. Hence $\bigsqcup_k \alpha_k^\nabla \in T_{\gamma+1}(p, n+p)$. But

$$\bigsqcup_k \alpha_k^\nabla = \bigsqcup_k \bigsqcup_m (\alpha_k, \pi_n^{n+p})^m \bullet (\perp, \ldots, \perp, \mathrm{id}_p)$$
$$= \bigsqcup_m \bigsqcup_k (\alpha_k, \pi_n^{n+p})^m \bullet (\perp, \ldots, \perp, \mathrm{id}_p) := \left(\bigsqcup_k \alpha_k\right)^\nabla.$$

(b) $T_{\gamma+1}$ is closed under composition and tupling. Let's see for instance composition: suppose $\{\alpha_i\} \subseteq T_\gamma(p,q)$, $\{\beta_j\} \subseteq T_\gamma(n,p)$ are chains bounded by say ε, μ in T_γ. Then $\{\alpha_i \bullet \beta_i\}$ is a chain bounded by $\varepsilon \bullet \mu$ in T_γ and hence $\bigsqcup_i \alpha_i \bullet \beta_i \in T_{\gamma+1}(n,q)$. But

$$\bigsqcup_i \alpha_i \bullet \beta_i = \bigsqcup_i \bigsqcup_j \alpha_i \bullet \beta_j = \left(\bigsqcup_i \alpha_i\right) \bullet \left(\bigsqcup_j \beta_j\right).$$

(iii) If γ is a limit ordinal, and $T_{\gamma'}$ is rational for all $\gamma' < \gamma$ then:

(a) T_γ is the colimit in **PosTh**$_\perp$ of the $T_{\gamma'}$, by the ω-sorted version of 4.4(a) and the fact that filtered colimits in **Pos** have set the colimit of the sets and order the colimit of the orders.

(b) T_γ is rational because if $\alpha : n+p \to n$ in T_γ, then α is in $T_{\gamma'}$ for some $\gamma' < \gamma$, and so is too α^∇, by hypothesis. □□

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